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Learning Mathematics Well

1 Introduction

A strong and athletic friend of mine jumped into the water. After nearly three minutes of wild but mostly wasted flailing he completed his 100 yards. It was another 20 or 30 seconds before he finished panting. That's one way to get through your math class: dive in and thrash wildly throughout the semester. An alternative is to develop the means to learn mathematics effectively. Would you rather calmly learn a section of mathematics in a short time, or struggle for a long time only to realize that you still don't understand? This whole document is designed to help you develop the means to learn math.

For best results choose a potentially helpful section and read the section or a few subsections of it. If you intend to take full advantage of this document then start from the beginning. If you are already in over your head and need help this minute, start with the Best hint and More quick hints (sections 3.1 and 3.2). If you just want a few suggestions on fine-tuning your study methods, start with the diagnostic tests (section 6) to find which section will be most helpful. Then try some of the suggestions (or modify them to suit your situation) . Afterwards you can read further and take advantage of more suggestions. Don't get overwhelmed by the length of this document.

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2 What to Expect

2.1 College differs from high school

- Homework might not be graded. It might not even be assigned. Nevertheless doing the homework is crucial to success. If the homework is graded be sure to complete the homework and turn it in on time. If it is not graded work a few problems of each type. If you get them all right go on to the next type of problem. Get help if you have trouble with a certain type of problem.
- Lectures move faster. Professors use fewer examples, skip supposedly easy parts of problems, and skimp on reviews in order to stay on schedule. As a result you have to complete your notes after class. (See section 3.7.) You aren't done with the lecture until you understand it.
- Your share of the responsibility for the class is increased. Professors might allow poor classroom behavior or shoddy homework on the assumption that what you haven't learned yet you'll have to learn the hard way.

- Prerequisites are important. Professors assume you know and remember content from previous courses. If you have forgotten something a professor is likely to ask you to look it up on your own, or to ask outside of class. It is likely that reviews will be cursory or even absent altogether.

- Learn outside of class. In high school you learned most of the material in class, and the role of homework was largely to practice enough to remember the material. In college many professors expect you to learn on your own. A professor might skip a portion of the material and ask you to learn it independently. Many professors present all the material, but go too quickly for you to learn it. In that case you will need to learn the material outside of class using your notes and/or the text. Expect to study two hours outside of class for each hour in class. For difficult courses this might stretch to three or four hours of study for each hour in class. This is why between twelve and sixteen hours of classes a week is considered full time.

- The competition is stiffer. Your current classmates were the more successful high school graduates. You'll have to study longer and/or more efficiently just to maintain your previous academic rank.

- The crowd is bigger. In a large class the professor might not know your name or even recognize your face. You will have to respond to announcements rather than personal messages.

- The initiative is yours. College professors might consider their job complete when they have delivered the material, graded the exams, and answered the questions. Specifically, a professor might not warn students about poor grades or remind them of due dates. Therefore you must take responsibility to know the syllabus, to meet due dates (Write them on your calendar.), to learn the material, and to know whether your grade is good enough. Quickly seek the help when your grades are unsatisfactory or you have trouble. (See section 3.12.)

2.2 Mathematics differs from other subjects

- Most math classes teach skills based on concepts. Unlike non-conceptual skills, practice without understanding will only get you to the most elementary level. Unlike a concept that can be memorized, a skill must be practiced. You will learn the most with the least amount of effort by practicing regularly. This means do the problems. Skipping daily homework and then cramming for tests is like preparing to play in a concert by practicing for hours on end, starting a day or two before the concert.

- Vague familiarity is inadequate. You must be able to do the mathematics.

  A typical math question requires that you calculate the correct answer without any external clues. Any single detail that you don't know could make it impossible to answer the entire question. Some details might be required for nearly every question on the test. Nor can you forget the big picture for focusing on the details. You will need the big picture in order to choose an approach, especially for the more difficult problems.

  It is difficult to continue studying once you recognize the mathematics that you're covering, but you must pass the recognition stage to do well. Be sure that you know every step in solving a problem. You cannot skip steps; with a small gap in your ability you will ultimitely falter.

- Mathematical vocabulary is important regardless of whether it is explicitly covered on the tests. You will fall behind during lecture each time words are used for which you have to struggle to remember the meaning, or deduce the meaning from the context. The extra work of decoding the nomenclature prevents you from understanding the significance of what is said. This becomes increasingly important in higher math classes.

- Nearly all of mathematics is logical and sensible. Massive amounts of information can be remembered easily when organized into a pattern. If I forget that $3 \times 5 = 15$, I can recover that information by calculating $5 + 5 + 5$ or $4 \times 4 - 1$ or some other equivalent computation. Mathematics is detail intensive, but if you understand the big picture you can safely forget many details and reconstruct them as needed.
- Many math problems require a comprehensive set of skills. A single problem could easily require a "big picture" perspective to determine what to do, recent and distant prerequisite material, reading comprehension, and an ability to complete every single detail without mistake.
- Math classes are cumulative. Everything that you fail to learn is likely to cause an ever-increasing amount of trouble in your future math classes until you learn it.
- Math classes demand an increasingly sophisticated understanding of the material. This is reflected in the progression of types of questions considered reasonable throughout the curriculum.

    1. Testing Facts. These questions are common in kindergarten through about third grade and later during the introduction of new areas of mathematics. These questions require that you quote a fact that you were told to memorize. For the question $2+2 =$ you simply write 4. There is no process, no debate, no room for error, no partial credit. You simply memorize the answer.
    2. Testing Procedures. From first grade through sophomore level college math classes many questions are procedural. You are required to apply a process that you have been shown and have practiced. You may have never seen the specific example, but you won't have any question about which procedure to apply or how the problem ought to be worked. You simply follow the procedure that you have learned. Perhaps you have never computed $3,126 + 6,922 + 1,556$, but without any thought you know exactly which process to apply and how to carry out the process.
    3. Testing Procedure Selection. From first grade through the second or third year of graduate school many questions are application based. With these problems you have to determine which process needs to be applied, and then you have to apply the process in order to get an answer. Determining which process to apply, however, is fairly straightforward since the problems are similar to ones you have already worked. These questions are often story problems or multiple-step problems. To solve $x^2 - 2x = 8$ you must first rewrite the problem in standard form, and then either factor or apply the quadratic formula (or complete the square, or search for rational roots, . . .). You must choose the correct process and then apply it. Likewise with the story problem: "Gina drives for 3 hours at 65 mph. She then lets Jack drive her car for another 2 hours at 70 mph. What is the total distance they go?"
    4. Testing Principles and Procedure Use. These problems are typically seen before the freshman year of college only as special projects, but are common thereafter. The question is to verify that you understand the procedures and what they accomplish and can use them to solve cocmplicated problems. These can be set up as involved story problems, long multiple-step problems, or problems requiring standard procedures to be modified before use. Although these problems appear unfamiliar, they can be solved using principles similar to those used for standard procedures. The goal is to verify that you understand the

techniques that you've learned well enough to solve problems unlike those that you've seen and practiced.

5. Testing Theory. These problems are seen occasionally as special projects in junior or senior level college courses and show up regularly in graduate school. These problems can not be solved using standard methods. New methods must be developed from basic principles in order to solve the problem.

Each type of problem tests a deeper understanding of the material than the prior type. Many good students typically master all the types of problem that have been expected of them in the past, but feel that it is unfair when first faced with a deeper type. With good preparation, however, it is possible to use mathematics creatively, and to know when to revert to fundamental principles to solve unusual problems.

- Math moves from concrete to abstract. In every field classes generally progress from the simple to the more complicated. In some subjects this progression relies heavily on learning the material with increasing detail. In some subjects the progression is based on learning additional principles to organize and explain the material better. In mathematics the progression is rarely in the direction of increasing detail. Usually it goes in the direction of new material, as when you switch from basic algebra to geometry, or in the direction of increasing abstraction, as when you switch from adding 2 apples to 3 apples (numbers as adjectives) to adding 2 and 3 (numbers as nouns).

2.3 Calculus differs from lower mathematics courses

Some students do well in mathematics until reaching calculus, at which point they stumble. Calculus has a different perspective than the previous courses. Understanding and adapting to these changes can help you succeed in calculus.

- Most prior math classes relied less on previous knowledge. In calculus you will be expected to know arithmetic, algebra, geometry, and trigonometry. Many calculus problems require proficiency in at least two of these areas, and some require that you know all four. Your weaknesses in any area will hinder you throughout calculus. If you have any mathematical deficiency, try to identify and correct it.

- Most math classes prior to calculus focus on numbers. Calculus focuses on functions. Be sure you understand functions and functional notation. Expect questions that ask about functions and/or their properties. Many final answers will still contain x (or another variable).

- Calculus is one step further removed from rote procedural methods. Previously you were shown exactly how to solve each type of arithmetic problem. You were shown the principles used to solve algebra problems as well as examples to illustrate these principles. Every problem that you were expected to solve was quite similar to one of the examples. In calculus you will see problems that look different from the examples, although the solution will use the same principles in a similar way. As you move into calculus the concept gains value relative to the process.

- Like algebra, calculus is computational and has the goal of solving real problems. Like geometry, calculus is theoretical and includes proofs. Depending on how the class is taught, you might have to use and understand the proofs or underlying logic as well as the methods. Unlike geometry, the proofs are often algebraic in nature, and some are rather long and difficult.
- Since calculus is both about theory and about practice, you will probably be asked to prove that you understand a problem by solving it in a particular inefficient way. Students who have already learned shortcuts or simple procedures that result in the same answer as a long, difficult, and tedious process can find it quite frustrating to be forced to use the latter. Nevertheless, you may need to do so to illustrate your understanding of a related issue.
- The pace, breadth, and depth are greater than in most prior classes.

2.4 Upper level mathematics courses

- Learn the vocabulary. Many advanced courses are built around a few precise definitions. You will do yourself a favor to learn these well enough to use them correctly without hesitation.
- Note the focus of the course. Many advanced courses are more about theory, process, and proof than about computing answers.
- One key to understanding a theorem is to determine what would happen if the conditions were violated. There are two main answers: That the theorem is still true, but it is beyond the scope of the class to prove under these conditions; and That the theorem is false with the changed conditions. Finding out why it is false with the changed conditions can lead to a greater understanding of the theorem, either for the purpose of proving it or just for the purpose of understanding it well and using it better. For example, there is a theorem that states: "Suppose that f is continuous on the finite interval $[a, b]$. Let c be a point where f attains its maximum. If $a < c < b$ and f is differentiable at c, then $f'(c) = 0$." What if f is not continuous? What if $[a, b]$ is infinite, or f is continuous on an open or semi-open interval? (In each of these cases then it is possible that there is no such point c.) What if $c = a$ or $c = b$? (Then the one-sided derivative could be positive at b or negative at a.) What if f is not differentiable at c? (Then the derivative would not be 0.) Knowing how things can go wrong will help you to understand how each of the required conditions contributes to the conclusion.

3 General Study Skills

3.1 Best hint

Practice!

The better your effort, the more you will learn.

3.2 More quick hints

- Actively participate in class. Try to figure out the professor's next step. Take notes. Do some of the work rather than just copying it all. Pay attention to the class, not your phone, your classmates, a bug in the room, or any other distraction [3].
- Review your notes shortly after class. Add explanations or examples where necessary. Identify what you don't understand, resolve the problem, and learn it.
- Do the homework. Start the homework right away so you have time to get help if you need it.
- Don't get behind. It's harder to catch up in a mathematics course than to learn the material at the pace at which it is presented.
- Study regularly. You'll learn more by studying an hour each day Monday through Friday than by studying for seven hours Sunday afternoon. (It's best to study daily or 5 days a week. At least study three days a week, with none of those days back to back.)
- Learn from your mistakes. Some mistakes are cheap and some are costly, but all cost extra when repeated [6].
- Test whether you know the material by working a few problems without using your notes or getting any other help.
- You haven't finished studying the material until you understand it.

3.3 Effective study practices

According to Sun Tzu, "victorious warriors win first and then go to war, while defeated warriors go to war first and then seek to win."

- Stay healthy. You will concentrate and study more effectively if you are healthy. You've already heard about the benefits of a healthy lifestyle with a proper balance of sleep, diet, exercise, prayer, relaxation, solid friendships, an optimistic outlook, and so on.
- Manage your time. If your life, schedule, and classes are overwhelming, the first two things to do are to succor your soul and to manage your time more effectively. Get a day planner. Use a day planner to help you get through your duties as efficiently as possible, and to help you determine how much time you can save for the extras in life. (Once you manage your time effectively in general, all these other study skills will be easier to master and more profitable. For extra help with time management please see your professor, student services, or get a book on the subject.)
- Choose goals. The world offers far too many options for you to take advantage of them all. Choose which ones to pursue with determination and which to pursue as opportunity arises. If you choose to pursue good grades, put in the effort required to earn them.

- Save study time. The rule of thumb is that you will require two hours of work outside of class for each hour in class. That's actually a minimum. To reach your potential you'll need to spend two hours for your easy classes and up to four for your difficult classes. Schedule enough time to complete your studies. You'll be much happier getting some bonus free time if your studies go quickly than struggling to find extra time if your studies go slowly.

- Study while alert. Your ability to memorize deteriorates when you are tired or stressed. Your ability to think clearly deteriorates even more. Therefore it's best to study all of your classes while you are alert. If you must study while drowsy, choose the subjects that rely most heavily on memorization.

- Avoid distractions. To reduce distractions study in a comfortable, quiet, plain room. Each distraction reduces your ability to concentrate on the mathematics, regardless of whether the distraction is planned or pleasant. While listening to music your attention is split between the mathematics and the music. Talking with friends, text messaging, watching TV and similar distractions are even worse. There are, however, three beneficial uses to music. First, sometimes you can use music to drown out noise that is even more distracting. This works best with music that doesn't catch your attention. Try elevator music. Second, if you must stay alert but don't need to concentrate, such as while doing work that is tedious but doesn't require much attention, you can use music to give your brain something extra to do to help you stay alert. Third, if you lack the mental discipline to do your work without the consolation of listening to music at the same time, you will be better off doing your work while distracted than skipping it entirely.

- Start well. What happens if you over-prepare for the first test? Perhaps you get a higher grade than you want. Perhaps you waste a few hours. You might decide to study less for subsequent tests. What happens if you under-prepare for the first test? If you want to maintain a good grade you'll have to do even better on the remaining tests. It can be very difficult to recover a good course grade after a bad test. For example, to get an A- in a class with five tests of equal weight after getting 60% on the first one requires an average of 97.5% on the rest. Until you know what is expected, it's better to be over-prepared. Another advantage to starting well is to minimize the stress of final exams. It is much more pleasant to go into your final exam knowing that if you get a C or higher you'll get at least a B in the class, than to know that you need an A or higher to get that B.

- Prepare for class. Obviously you should complete the recommended readings and assignments and bring the appropriate supplies to class. Beyond this, you should set aside distractions so that you're ready to concentrate. Also write down your questions so that you remember to ask them during class. Review prerequisite material if you are out of practice.

3.4 Focus

Determine the focus of the course. Although it seems that all math courses are focused on solving certain types of math problems, a more nuanced determination of the goals can be helpful. Will you be expected mostly to repeat the types of calculations demonstrated in class? Will you be expected to choose the correct calculation for a context (such as with a story problem)? Will you be expected to distinguish between apparently similar cases that require different approaches? Will you need to understand why a method works, or just be able to get the correct answer? Some professors primarily demand high proficiency with standard problems, others demand competency with unusual problems. Some professors use proofs in class primarily to illustrate a point. Others expect you to know and understand the proofs. Determining the focus of the course will enable you to align your efforts with the course objectives.

3.5 Reading the text

- Don't just read the text. You have to actively study to learn from it.
- Read (or skim) the text before the lecture on the corresponding section. Try to learn at least part of the lesson if the text isn't clear. One option is to focus on the major points. It might be easier to understand the major points by ignoring special cases or alternatives to the primary method or by glossing over the intermediate steps in the calculations. Another option is to focus on the details. Figure out how each step leads to the next in the example problems while ignoring the point of the calculations. A third option is to learn the vocabulary words. Learning part of the material ahead of time will make the lecture easier to understand and will also prepare you to ask good questions. (See page 18.)
- Read (or skim) the text after the lecture. This time through you should understand the text. Make sure that you also understand how it relates to what was covered in the lecture.
- Most math texts consist of both prose and mathematical equations and formulae. Be sure to read them both. The prose is usually better for explaining when to apply the method, under what conditions it works, why it works, and hazards to avoid, while the equations often illustrate the method more clearly by working an example.
- It is easy to gloss over restrictions and qualifying statements (such as "If all the coefficients are integers" or "For all positive numbers x"). Feel free to do so on the first pass if that helps you understand the material but eventually you should learn these conditions. It's valuable to know why they are needed.
- It's much easier to understand the text if you read it in several phases [1].

  Preview: Spend roughly ten to fifteen minutes previewing the material. Scan through the section and read the subsection titles, main points, vocabulary words, and so forth. Look at the homework problems just to see what types of questions you should be able to answer based on this section. Determine the main point of the section and what major ideas it uses. It's not necessary to understand more during this first phase. A missionary to a foreign country introduced and illustrated his point with a long tale about a caterpillar. Several minutes into the tale, when the caterpillar truned into a butterly, the translator realized that the missionary had not been using 'caterpillar' as a synonym for 'tractor.'

  Read: Actively read the section. More ideas are given below.

  Review: After you have read the section review the material. Refresh your memory as to the main points so that you don't lose them amongst the details.

  Do the homework: (See 3.10.)
- To get the overall picture it might be helpful to ignore calculations. As you read through the text searching for the big picture, when you reach the calculations, jump over them and say

to yourself, "We do the work and we get the answer ..." and then continue. Actually do the calculations at another time when your focus is on understanding the details or the entire picture rather than just the big picture.

- When following calculations in the text:
  - Verify that you know how to work each step.
  - Work the problem on your own. You may look at the text for help, but be sure that you understand each step before moving on to something else.
  - Work a slightly dissimilar problem to verify that you understand well enough to handle small variations in the problem.
  - Write down the mathematical process applied at each step.
  - Write down the reason that each step was taken.
  - Fill in any gaps. Sometimes it takes two, three, or even more steps to get between what the book wrote as consecutive steps.
- Pause as needed to reflect. Think about what the material means, how it relates to previous ideas, and what are the consequences [3].
- Sometimes it is more convenient to take notes from the text and study from the notes. This is especially true if you only need to study a few portions of the text.

3.6 In class

- Warm up before class. If possible, take a few minutes before class and review your homework that is due. Skim your notes or the section in the text. Remind yourself of what you are doing in your math class so that your mind will be ready to move forward as the class starts. Ask a classmate if you had trouble with the homework. Regardless of whether this allows you to avoid the horror of asking a question during class, your brain will get into gear and start warming up for class [1].
- Be prompt. Have your paper and pencil out and ready, your homework handy, and any questions at hand by the time class starts. You will prefer to get a good start on the class each day, rather than lagging from the outset and then having to catch up.
- Think through the process as you are able. Don't merely wait for the professor to work each step. Work it yourself first. Even if you don't have time to do the work, at least decide what the next step should be done. When the professor does what you expect this verifies that you understand the problem and when he doesn't you need to learn why.
- Ask questions. Rarely does a professor describe everything perfectly. If you don't understand something ask about it. Professors can address the point of confusion much more effectively if you bring it to their attention.
- In math classes the importance of material is only weakly correlated with whether it is written on the board. Most computations are written on the board because it is very difficult to work them out and communicate them orally. Many professors will elaborate on the work with oral explanations. These oral comments can be more valuable than the written work. Be sure to note all of the important points, whether written or not.

- Consciously decide how much detail to write. If you record every comment you will spend your entire time writing and won't have much time for thinking through what has been said. If you thoughtfully consider all that is said you won't have a decent record for review when you need to study. Therefore you should sometimes write down just the key steps of routine calculations to save time. Specifically, if there are portions of the lecture that you can recreate on your own, or that you already know, or that are explained clearly in the text, you might not need these in your notes, or if so, only in brief outline form. Write enough to record everything important. Beyond that write as little as possible in order to free your brain for thinking.
- Find a partner that you trust to take good notes. Then on alternate days one of you can take thorough notes while the other follows the lecture without distraction. After class one can share a deeper understanding while the other shares more complete notes.

3.7 Using notes

- Taking good notes is not enough. Remember that if you take good notes you have provided yourself a good tool for learning the material. You haven't learned the material until it moves from your notes into your head.
- Save time in class by using a personal shorthand. If you use a word often you can invent a personal symbol for the word and use it in your notes. Be sure to define your symbol before you use it lest you forget what the symbol means when reviewing your notes later. For example, you might write "$\bar{P}$ = polynomial" and then use "$\bar{P}$" in place of "polynomial" for the rest of the day.
- Organize your notes. One way to organize your notes is to use two columns. Put examples in the left column and other notes and comments in the right column. Another possibility is to use an outline format, putting the main points against the left margin, indenting the subpoints, and indenting the subsubpoints farther. You could use different colors or special symbols (arrows, stars, boxes, underlining, etc.) to mark each type of content.
- Organize all of the class material. Your notes will not be very useful unless you can find what you're looking for. One good way to organize your notes requires that you use a three-ring binder, preferably one dedicated just to your math class. Divide the binder into five sections, one each for handouts, class notes, homework, returned exams and quizzes, and summaries. In the summary section put a glossary of terms, brief explanations of how material from one section relates to material from other sections, and those things that you learn that are pertinent to several different types of material [1].
- Complete your notes. As soon after class as convenient go back over your notes and write them out completely. Fill in any steps missing from a computation. Write explanations for parts that are unclear. Review prerequisite material as needed. If you wrote something in summary or outline form write it out in complete sentences. It is easy to understand a skeletal description while the event is still fresh in your memory but hard to understand it later. Ultimately you need to understand and remember what is written in your notes.
- Some people prefer to leave blank spaces in their notes to leave space for filling in the gaps and completing the notes. Others prefer to rewrite their notes on a fresh sheet of paper. The latter is more work, of course, but rewriting and reorganizing notes can be an effective learning tool.

- When the professor works an example problem in class, work a similar problem on your own. Doing so will help you spot whether your notes are inadequate, as well as what is general for that problem type and what is specific to that example.
- Spend extra time on any material that is especially difficult [5].

3.8 Group work

Most people learn mathematics more effectively when they discuss and do math with a small group of other students.

- Choose good partners. A friend is not necessarily a good choice. You might distract each other too much. It's easiest to collaborate with students who will get grades similar to yours. People with much higher grades might explain material at a level that is too advanced, get frustrated by spending too much time studying and explaining what they already know, feel that the group moves on before mastering an idea, or end up feeling that they are giving much more than they're getting out of the study group. The corresponding problems arise when working with people heading toward much lower grades. Make sure study partners are available at similar times or scheduling will be difficult. Group study sessions should be scheduled once or twice a week, regardless of whether there is an upcoming test. You might want to contact members of your group with quick questions between your regular meetings.
- Great benefits come from going over difficult homework problems together. Those who got the right answer can explain it to the others. If no one gets the correct answer you should try as a group to solve it.
- You'll gain much from cooperative problem-solving. It's common for a few people to know how to solve a problem between them, even if none can solve the problem alone. This can happen if they all have misunderstandings in different parts of the problem, but it can also happen if the people critique and correct one another's attempts and together work toward the correct solution logically. While this can be done individually, the group interaction makes it easier.
- Try this process to be sure that you understand the material better: One member of the group works a problem. Someone else asks "How come that's true?" at each step. Another person asks "How come that's helpful?" at each step. Consistently explaining what you're trying to do and how you're doing it keeps the details in perspective and draws attention to errors. A designated critic can also help as a group works to solve a difficult problem.
- Even if no one in your group understands a problem you still gain the fringe benefit of being able to ask in class knowing that you're not alone with the question.
- After a group study session in which a classmate explains a problem to you, make sure that you learn to work it yourself.
- Remember that you can learn the material by explaining it to others or by receiving an explanation.
- Share note-taking duty in class. (See section 3.6.) A group can help with back-up plans for notes in case you miss class.

- Before the test have each person in the study group present a summary of the material using an organizational principle different from that used in class. This will help you to see connections that you might have overlooked.
- Have your study group create practice problems before a test. Since the questions that you ask yourselves are flavored by different perspectives, you get a valuable diversity of problems.
- Peripheral conversations can be very valuable. Discuss anything of interest to you that relates to the material in class. For example, if a calculus homework problem is to maximize a profit you might be tempted to discuss: how the problem relates to your goals of making money; how it is too simplistic to relate to real commerce; how to deal with an unknown demand function; how to combine short with long-term goals or how it could (or couldn't) be modified to improve the Cubs batting order. At other points in the course you might discuss the philosophical concept of infinity; whether the universe is truly continuous; whether it is sensible to claim that $\infty = -\infty$; whether this or that problem are better analyzed using Reimann sums or anti-derivatives. The truth is that math relates closely to many other areas. By discussing these relationships you'll gain insight into how math relates to the rest of the world, you'll see more of the power and applicability of mathematics, and better understand its ramifications. Although these conversations might not help you immediately, in the long run they can contribute greatly to what you learn in the class.

3.9 Calculator work

- Know the class policy for calculator use on tests and homework.
- Verify that your calculator is suitable for the class.
- If calculators are allowed on the test, be sure to review all calculator procedures pertinent to the test. It is easy to forget the key sequence required for seldom-used procedures.
- Know how to work problems without a calculator, if necessary. Know the principles behind the problem. Most professors ask a few questions to distinguish between those that understand the material and those that merely know how to punch the right buttons.
- Verify that the calculator is in the correct mode. For example, it is common to incorrectly work trig problems using degree mode, especially if your math class uses radians and your physics class uses degrees.
- Know how your calculator will handle order of operations. If you type in $1 + 2 \times 3 =$ do you get 9 or 7? Do you get the same result if you type $1 + 2 = \times 3 =$?
- On the test verify that your calculator has given a reasonable result before you write down the answer.
- You may be eligible for partial credit if you write on the test both the formula and the result of the computation. It is easy to push the wrong button, get the wrong answer, and get no credit.

3.10 Doing the homework

- Work enough homework to:
  - Verify that you understand how to work the problems.
  - Know how to handle the full range of problems of the type covered in the section.
  - Understand the range of applications for the methods covered in the section.
  - Practice enough that you can work the problems with reasonable speed and reliability without help.
  - Practice enough that you will remember how to work the problems later. (This is easier if you work a few problems on each of several days rather than all the problems at once.)
  - Gain confidence that you can work this type of problem.
  - Get a good grade on any assignment to be graded.
- Work extra problems. If you don't learn the material adequately before completing the assignment, find some extra problems to work. If you work nearly all the problems in the text, you can work problems from another text or you can make some up. Ask the professor if you can't find enough problems in the text or in the library.
- Work each problem neatly and professionally. This makes it much easier to study from your homework.
- Take notes as you do your homework. Make a note of important formulas. Write down hints and tips for yourself. The notes that you take from working your homework can be just as important as your class notes [5].
- Be diligent. If you come to a problem that you don't know how to work, and you've checked your notes and the text and all your other sources that are handy and you still don't have the answer, then work ten[1] more minutes on your own. (See section 5 for suggestions on how to approach the problem.) If you get the answer, you'll be proud of your accomplishment, and if not, you'll know why the problem was difficult [1].
- Be economical. If you exhaust all of your options to find a solutioin to a problem and still don't know how to do it, then set it aside and go on to something else before you waste all of your study time (and patience) [1].
- Write the truth. Don't arrange your computational aids so as to obscure the truth. For example, don't write: $x^2 + 3 = 12 = x^2 = 9 = x = \pm 3$. In fact, if $x^2 = 12$, then $x^2 \neq 9$. Likewise, if $x = 9$, then $x \neq \pm 3$. A much better way to write the problem would be: $x^2 + 3 = 12 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Here the arrows show the progression of the logic without claiming equality. If you organize your work vertically it makes sense to skip the arrows.
  $$\begin{aligned} x^2 + 3 &= 12 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$
  If you need comments to help with the work, put those comments to the side, or between lines of true statements rather than intermingled with the true statements. Some ways to include

[1] Make this fifteen minutes in a calculus or linear algebra class, and thirty minutes in classes that have one of these as a prerequisite.

extra comments for the first step include:

$$\begin{aligned} x^2+3-3 &= 12-3 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned} \qquad \begin{aligned} x^2+3 &= 12 \\ -3 & \quad -3 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned} \qquad \begin{aligned} x^2+3 &= 12 \\ x^2 &= 9 \qquad (-3) \\ x &= \pm 3 \end{aligned}$$

- Check your homework.

  From the text: If the answers to your homework are given in the back of the book, don't check your answer to a problem until after you have completed the problem. If you know the answer before you start, that might be enough of a hint for you to complete the problem correctly even though you don't quite know how to work the problem. If you make a mistake, be sure to rework the problem and also to learn the correct approach. Just fixing the error is not enough.

  With classmates: As long as you don't violate class expectations for working with others this can be a great way to check your work. Clearly if you and a classmate have different answers then at least one of you is incorrect. You can learn a lot by checking your work, and then discussing with your classmate the correct way to work the problem.

3.11 Verifying your knowledge

- If you refer to your notes or get any other form of help while doing the homework, then work a few problems a day or two later to verify that you know how to do them by yourself.
- Starting several days before the test work two or three problems from each section that are like the hardest ones on the homework. Homework problems suffice but it is better to choose new problems. If you can work these problems you should be ready for the test (unless some of the simple problems test specific information or methods that weren't required for the complicated problems). If you can work the difficult problems don't waste your time working more of the easy problems. Go on to a different section. If you can't work the difficult problems start practicing with the easier problems and work up to the challenging ones.
- One problem with homework problems is that the section gives you a big clue about how to work the problem. This particular clue will be reduced during the test since you will have questions from several sections. To verify that you can work the problems without this extra clue be sure to work problems without knowing what section they're from. The easiest way to do this is with a study group. One person can read a question to the group without disclosing the section that it's from. Another way is to write down several questions, and then scramble them up (such as by drawing them at random from a well-mixed pile) before working them [5].
- Take a practice test. A well-executed practice test is the best tool for finding what you still need to learn. It can indicate whether you can work problems quickly and accurately enough, without any help, to get the grade that you want. First, get a test with questions similar to what you expect to see on the real test. You might have practice tests in the text or be able to get an old test from another semester. If not, make your own test by selecting some homework questions, other problems from the text, and/or asking a study group to help make up questions. Second, work the test in the same amount of time that you'll have available and

using similar tools (no notes, text, or study partners). Third, grade yourself. Fourth, study the things that you missed.

- In addition to indicating whether you know the material, self quizzes can be an effective study technique [2].

3.12 Getting Help

You will improve your math skill and gain self confidence if you solve all of the problems on your own. Persistence is valuable. You've probably heard that Thomas Edison claimed that genius is 90% perspiration and 10% inspiration, or that he built about 1000 different lightbulb desgins before finding one that would work. Nevertheless, there's no point banging your head against the wall or reinventing the wheel. When you realize that you need help, get it soon (definitely within a week) [5].

- Don't be ashamed. Admiral Rickover introduced nuclear propulsion to the U.S. Navy submarine force. He taught one class in which a cadet who was having trouble learning the material asked Rickover to provide a regular study session with a tutor. Rickover insulted his intelligence but arranged the study session. Every cadet in the class showed up. You are rarely alone when you have a question, so asking it is usually beneficial both to you and the other students. (By the way, few American professors outside the military will insult you for asking a question.)
- Know where to look for help.
  - Ask the professor in class, during office hours, by e-mail or make an appointment.
  - Ask classmates.
  - Ask a math major or someone else that has already taken the class.
  - Get help from the tutoring center. (NPU offers a fixed amount of tutoring for free each week. To save both time and money and to increase your chance for success arrange for regular tutoring rather than test-cramming tutoring.)
  - Join a study group and ask questions there. (Start the study group if necessary.)
  - Ask whether there is some sort of math clinic, math study hall, or extra instruction period that would be helpful for that class.
  - If the text is confusing, try finding a supplemental text in the library, the bookstore, or on-line. The professor, the librarian, the book retailer, and on-line book reviews might have good suggestions.
  - Try an on-line supplement. Many public websites explain mathematics. Many texts come with on-line supplemental material.
  - Ask the professor to recommend sources if you can't find the help that you need.
- Get help effectively.
  - Start early. One timely question is worth a dozen a week later.
  - Use your time effectively. Have your questions ready when you meet with a tutor, with a study group, or with your professor outside of class.

  - When working with a tutor, be sure that the tutor answers your questions rather than something else. If the explanation doesn't make sense, insist on a better explanation.
  - If your tutor suggests focusing on issues that are more fundamental or prerequisite to your specific questions, consider the suggestion. Resolving one fundamental misunderstanding might help with several apparently unrelated questions.
  - If you run out of time, make arrangements to meet again.

- Ask good questions. Typically a precise question is better. If you can direct the professor (or tutor) to the exact problem you are much more likely to get that problem fixed than if you only direct the professor to the ballpark. Diagnosis precedes treatment.
  - "I didn't understand the homework." Indicates that there's a problem. It gives no clue about how to fix the problem.
  - "How do you do number 17?" This focuses on the specific math problem, but not the part of the problem that caused you trouble. Consider the improved questions: "I got problem 15, but not 17. What's the difference?" or "I got problem 15, but not 17. How do you handle the extra term?" or "I couldn't set up problem 17." Each of these versions of the question are better because they begin to focus on some aspect of problem 17. An excellent question might be: "Here's what I tried on question 17. What went wrong?" This brings the professor's attention to precisely where you went wrong.
  - Many students ask "How did you do that?" in response to some work on the board. Be sure to put the professor's attention on your exact question. First, direct his attention to the "that" by saying "How did you get from your third line to your fourth line?" Next, indicate what you mean by "How?" You might be asking, "I don't know what operation you performed. What did you do?" Or you might be asking, "It looks like you made a mistake. Will you check your arithmetic or algebra? And if it is correct, you'll have to guide me through it step by step." Or you might be asking, "Why do you know that those things are equal?" Or you might be asking, "Why do you think that is a useful thing to do?" or "How did you know to do that?" In some cases, especially when introducing a new element, (such as when moving from $x^2 + 2x = 3$ to $x^2 + 2x + 1 = 3 + 1$) the question might be "What gives you the right to do that?" or "How on earth would I know to do that" or "How do you know what new element to use?" A perfect answer to the wrong version of "How did you do that?" isn't helpful.
  - Questions are also better when they ask about a deeper level of understanding. Good questions might ask how the current material relates to prior material or to something from another class. You might ask about an extension, such as whether a method will work in a different situation, or what would you do if the condition for the method failed.

- Respond to the help. Write down the answers to your questions. Make sure that you learn the answers. Hearing them might not be enough. One of my students asked where one of the terms came from on a certain line of the computation on the board. I answered that it was the middle term that results from multiplying two binomials. This happened more than twenty times. What's the point of the question if you're not going to learn the answer?

- Don't rely on your tutor. Tutors can help with a lot of things but you'll face the exam alone. Make sure that the tutor inspires and equips you to proceed rather than giving your brain a chance to atrophy.

4 General Test-Taking Strategies

4.1 Managing your time

The whole goal is to get the most points possible in the allotted time. Sometimes this is accomplished by answering each question in turn and not wasting any time planning your approach. More commonly, however, you are better off working short questions before long questions. Below is a good way to arrange your work.

1. Start by writing your name and reading the instructions.
2. Preview the test to see how long it is and whether the questions at the end look different from the questions at the beginning. Use this information to get a rough estimate of how quickly you must work. On an 8 question test to be completed in 50 minutes you have about 6 minutes per question.
3. Quickly work through the test from beginning to end. Complete the easy questions and skip the difficult ones. Indicate whether each question is complete, to be completed on a second pass, or to be saved for a third pass. [2]
4. Start from the beginning again and work the problems that you have saved for a second pass. Skip any that take too long and relabel them for the third pass.
5. Go back again and work the remaining questions, skipping the most difficult of them until completing the others. Continue until you complete the test or you run short of time.
6. As time runs short quickly decide how best to finish the test. Take into account your confidence in your answers and the manner in which your professor assigns partial credit. Good finishing strategies might include:
   - Checking your work. This is especially valuable if you aren't confident in your work, or if you will suffer major penalties for small errors.
   - Outline a solution procedure for the remaining problems. This is especially valuable if you can pick up most of the points by showing that you understand what must be done, even if you don't show that you know how to do it.
   - Rushing to solve one more problem. Do this if you're confident in your work, but don't expect much partial credit for grossly incomplete answers.

4.2 Easy points

Be sure to get the easy points.

[2] An alternative strategy is to start with the difficult problems so that you can mull them if you get stuck. (See section 5.2.8.) This strategy requires a careful balance to ensure that you finish the easy problems and also benefit from the mulling.

- Finish all the easy problems.
- Follow the class expectations for showing your work and formatting your answers. Simplify, include units, use the correct number of significant digits, or give exact answers as needed.
- Verify that your answers are appropriate. (See section 4.6.)
- Don't lose points twice (or worse). Learn from your mistakes before the next test. (See section 10.3.)

4.3 Documenting your work

Leave well-organized readable work to document what you've done. Few professors will carefully search sloppy and disorganized work for traces of a line of reasoning that merits partial credit. If you clearly indicate your thought process then you are more likely to get partial credit for a logical approach or for portions of the problem done correctly. This entails defining variables and perhaps also writing a phrase or sentence to explain what a computation was intended to accomplish.

If you make a mistake fix it as quickly as possible. Usually crossing out work is faster than erasing it. Crossing out work with one or two lines is faster than blacking it out entirely. An advantage is that if you later realize that your original work was correct, you will be able to recover what you had written.

4.4 Outline

If you know what you need to do but do not know how to accomplish it, then write an outline of how to solve the problem. This might might earn substantial partial credit. It might also lead to a complete solution.

4.5 Guessing

- On multiple choice tests: Know whether there is a penalty for incorrect answers. If there is a penalty then you should only guess if you can eliminate a few of the options. If there is no penalty then you should guess on every question. As a general rule keep your first guess.
- On compute-the-answer tests: Know the role of the work and know about penalties for wrong answers. If you will be penalized for wrong answers you should only guess with caution. If a correct answer without the corresponding work earns little credit (or if it raises suspicion of cheating) be sure to give some explanation of your guess.

4.6 Checking your work

One test contained the following problem: *The Dead Sea is 1,350 feet below sea level. Jerusalem is 2,500 feet above sea level. Federal regulations restrict the slope on certain classes of highways to no greater than .07. What is the minimum possible horizontal distance between Jerusalem and the Dead Sea if Israel can build a straight road from the Dead Sea to Jerusalem that obeys the U.S. grade limit?* Amongst the answers given were these two: “Jerusalem would have to be .378 feet from the Dead Sea.” and “36,428 miles.” Neither of these answers is correct. Regardless of whether the students answering this question made relatively minor mistakes, or whether they badly botched the entire calculation, they should have noticed that the answers were not reasonable. Consider the first of these answers. Going from 1,350 feet below sea level to 2,500 feet above sea level over a horizontal distance of less than a foot. This is a huge cliff! This is too steep for any vehicle to climb, much less appropriate for a highway. Using a more formal argument, note that .07 of .378 feet is much less than 1 foot, even though we need to gain nearly 4,000 feet of elevation. The answer is much too small. Now consider the second answer. Using common sense one might wonder where you would put a road 36,428 miles long, since it would reach one and a half times around the world. Even if the student doesn’t know the circumference of the world, the student might know that the U.S. is about 3000 miles across (excluding Alaska and Hawaii) and that the roads that climb the Rockies don’t cross the whole country and then jut out into the Atlantic Ocean. Checking the problem mathematically, we see that 7% of 36,428 miles is 2,550 miles, much more than the desired 4,000 foot elevation gain. (Even if you have forgotten how many feet are in a mile, I hope that you know that a mile is well over two feet long.) Knowing that their answers were unreasonable, these students could have checked their work and they might have found and corrected their errors. Even if they hadn’t found their errors, they might gain some points (and some esteem) by noting on the test that the result is clearly incorrect.

The value of checking your work varies depending on the reliability of your work and the distribution of partial credit. Often a solution can be checked much more quickly than it was computed in the first place. When an answer is wrong, the fastest but least reliable way to find the error is to look over your work. You might find the error, or you might just follow your work through the same error again. The more reliable method is to rework the problem from scratch without looking at your previous work until you have finished the problem (or reached a point at which you have a different result).

Verify that you have answered the correct question. In some cases the majority of the work yields information from which it is easy to complete the answer. Don’t miss the question after doing the hard part. (For example: A right triangle has two legs of length 12 and 16. What is the length of the perimeter? The hard part is to use the Pythagorean Theorem to calculate that the third side has length 20. In the relief of completing the initial calculation it’s easy to forget to add the three sides to get a perimeter of 48.) To be sure you answered correctly, review the question and your answer to verify that you have a reasonable answer to the question written according to class standards.

4.7 Re-creating knowledge

Math tests are special in that it is feasible to reconstruct much of the knowledge needed to do well on a test. For example, suppose that you need to compute $(x^2)(x^3)$. You're fairly certain that the answer is either x^5 or x^6 since you remember that there are two rules about exponents: one is to add the exponents and the other is to multiply the exponents. Unfortunately you can't remember which pertains to this situation. There are two ways to recover that information. The better way is to refer to the meaning of the symbols. It goes like this $(x^2)(x^3)$ really means $(xx)(xxx)$ and this can be simplified as follows: $(xx)(xxx) = xxxxx = x^5$. The other way is to create an example. Suppose $x = 3$. Then $x^2 = 9$ and $x^3 = 27$ and $9 \times 27 = 243$. On the other hand $3^5 = 243$ but $3^6 = 729$. Thus we suspect $(x^2)(x^3) = x^5$ for all positive numbers x and choose that as the best answer. Note that this method might not work so well on the problem $(x^{2000})(x^{3000})$ since you don't want to write out 2000 x's or the following 3000 x's, nor will most calculators handle 3^{3000}. If you are reasonably sure that $(x^a)(x^b) = x^{a+b}$ or that $(x^a)(x^b) = x^{ab}$, then you can check out the rule by example using small numbers for a and b, or you can still use the meaning:
$$x^{2000} \times x^{3000} = \overbrace{x \cdots x}^{2000 \text{ times}} \times \overbrace{x \cdots x}^{3000 \text{ times}} = \overbrace{x \cdots x}^{5000 \text{ times}} = x^{5000}.$$

All of mathematics follows deductive logic, so any of the rules can be reconstructed. While studying you should decide which rules to remember and which you would rather reconstruct. Base your decision on how hard the rules are to remember, how easy it is to reconstruct them, how much time you'll have on the test, and how much you are willing to memorize before the test.

4.8 Poise

Keep your cool. If you get stuck on a problem, don't panic. Panic will only hinder your ability to continue. Here are some ideas to reduce the likelihood that your brain will freeze on you.

- Prepare for the test by learning the material. "Fore-warned is fore-armed." If you are confident that you know the material and can work the problems then you don't have to worry about a stray hard question on the test. Do your best on the rest, and then return and do your best on the stumbling block. See section 5 for ideas on how to solve difficult problems.
- Prepare for the test mentally. Arrive early enough to get into the right mindset. If you have a tendency to panic during math tests you might profit from some regimen to preserve your senses. Any of the following can be helpful:
  - Prayer: If God is for you then no evil math question can stand against you.
  - Meditation: Clear your mind of distractions and focus all your attention on the problem at hand.
  - Ritual: Familiar motions can be comforting. Develop a little ritual that will help you calm down when you're nervous.
  - Practice: Take a practice test so that you become more familiar and confident about the test. See section 3.11.

  - Relax: If something shakes your confidence during the test, take a moment to relax and regain your composure. Your composure is more valuable than a few seconds.
  - Consider the consequences: It's unlikely that one bad question on one test will ruin your life. Even one bad test won't ruin your life. Even one bad class won't ruin your life. When you put the test in perspective you might find it easier to do your best.
  - Do something: Perhaps the least effective problem-solving strategy is to stare at the math test and repeat to yourself "I can't do this." Find something that you can do that will have at least some possibility of leading toward the right answer.

- Prepare yourself. Get a good night's sleep before the test. Make sure that you're comfortable. Use the restroom before the test. Bring a bottle of water and a sweater. That way if you get thirsty or the room gets cold you can still make yourself comfortable. Wear comfortable shoes and clothes. Bring some tissue in case you get the sniffles. If the seats are hard bring a little pillow to sit on.

- Prepare your tools. Bring two calculators to the test so you don't have to worry about bad batteries. Bring extra pencils and perhaps a pencil sharpener. Bring scratch paper. If you are well-equipped you won't suffer from equipment failures.

- Consider the possibilities that lead to a difficult question on a test.
  - One possibility is that you have misunderstood the question. Read the question again. If there is a part that seems ambiguous get clarification from the proctor. Verify that you intend to answer the actual question rather than a more complicated one.
  - Another possibility is that it is a difficult question that will challenge your classmates as well. Keep your cool. Save it for later, then give it your best shot.

4.9 Help

Feel free to ask questions during the test unless you are explicitly told not to. The proctor is likely to answer questions about the meaning of a question, especially if the wording is ambiguous. If you know all but one part of a problem, tell that to the proctor and offer to trade some partial credit for a hint. It is the proctor's responsibility not to answer inappropriate questions. If you're not sure, ask.

5 Problem-Solving

Two of the students in a calculus class that I taught were starters on the football team. In the calculus lab one day they had to work a difficult problem. They tried one approach which took about five minutes to complete. It didn't work. They then stared at the question for about 30 seconds, and then gave up and asked for help. I gave them a hint but they had already given up and were unwilling to try again without explicit instructions. Initially I was puzzled that two respected starters would give up so easily since football teams value dogged persistence. Then I realized that they might not have had any alternatives to try.

No matter how good your education and how complete your training, you will probably face a problem that you haven't been taught how to solve. Your best chance of solving these new types of problems is to have some strategies to guide your thinking so that you can approach the problem effectively.

Part of mathematics is learning specific methods to solve specific problems. During World War I American troops were taught four steps for hand-to-hand combat: jab the chest with the bayonet, upper-cut the chin with the rifle butt, jab the face with the rifle butt, slash the head or shoulder with a downward stroke of the bayonet, repeat. Presumably this was an effective way to fight the expected opponent but it would be absolutely useless against a medieval knight, even an unarmed knight only wearing armor above his waist. An inflexible soldier fighting such a knight would have continued the futile assault until being stopped by exhaustion, and incidentally, earning a dunce's cap to wear to the next battle, since there are a number of obvious ways to win such a duel.

We turn our attention to widely applicable modes of thought for those challenging problems that you haven't been specifically trained to solve. Many effective general approaches to solving problems consist of minor variations on Polya's four-step process: Understand the problem, devise a plan to solve the problem, carry out the plan, evaluate the results. This may seem to be too general to be helpful [5]. On the other hand, even a vague outline beats staring at the page after getting stuck. The bulk of this section explains and augments Polya's method.

5.0 Attitude

Polya's method covers how to solve problems but skips how to prepare oneself to solve problems. Following Sun Tzu's advice (see page 8), we arrange for success before starting the problem.

5.0.1 Optimism

Everyone knows the story of the little engine that could. If you expect to succeed then you are more likely to.

5.0.2 Determination

"When the going gets rough, the rough get going." "Rome wasn't built in a day." Our culture is filled with stories and proverbs that say don't give up. Keep working on a problem even if you encounter an obstacle.

5.0.3 Favor ideas

If one idea to solve a problem won't work, the first impulse is to discard the idea. Don't. Instead look for a way to make the idea work. This might result in a workable idea. (As an experiment several engineers were asked to identify the shortcomings of proposals for a difficult problem. They were later asked to solve the problem themselves. Another group was asked to critique the proposals by indicating what it would take to get the proposals to work. They were later asked to solve the problem. The engineers from the the second group had better solutions [4].)

5.0.4 Flexibility

If you have trouble with a problem consider a broader set of options. Any single component of your proposed solution, if it is incorrect, and if you refuse to abandon it, becomes an insurmountable stumbling block.

A lens-maker was polishing a mirror to be placed in the Hubble telescope. Partway through the polishing regimen the lens-maker tested the mirror to see how close it was to the correct shape. The test results differed from the results from the grinding machine. The lens-maker decided that the grinding machine was more accurate and continued grinding the mirror. Every single test that was conducted disagreed with the grinding machine's readings. After the Hubble telescope was in orbit it was determined that the mirror had the wrong shape. An inquiry discovered that the grinding machine used a laser to measure the Hubble's mirror. This laser was supposed to reflect from the front surface a guide mirror but the reflective surface had been scratched off and the laser was instead reflecting from the rear surface of the guide mirror. This tiny error counteracted the tremendous effort made to create a great mirror.

On the other hand, once I accompanied my father to the hospital where he had just a few things to do before we continued on our way. He picked up some laboratory results, read them, commented that they weren't what he had expected, and then proceeded to prescribe the treatment that had already been planned. I asked why he didn't change the treatment if the lab results suggested something different. He said that the lab results were not enough to alter the judgment based on the other symptoms. In this case that was true.

This world is filled with incomplete and inconsistent information. When it's time to act, make your decision based on the best information available. If new information arises you might be able to improve upon your initial decision.

5.1 Understand the problem

The suggestions here are focused on story problems, although with minor modifications they are useful for understanding many types of problems throughout life. Each specific recommendation can be helpful. Each one takes time, however, so you are better off skipping those that don't help much until you reach the point where you need the extra help. You should choose a few of these suggestions and routinely apply them to story problems. As you get more skilled with the story problems you might begin to do some of these suggestions in your head or to almost instinctively decide to follow one rather than another. That's good. When you run into difficult story problems, though, remember to try every one of these suggestions, explicitly, and write down the results, before giving up.

5.1.1 Goal

Carefully identify the goal. What are you trying to determine? How will you know when you have it? What will the units look like [5]?

5.1.2 Nomenclature

Some problems are stated using nomenclature, technical vocabulary. You must be able to identify when a word takes on its ordinary meaning and when it has a special meaning. The technical meaning of a word might be very precise and convey exactly what is necessary. For example on an old sailing ship a gallant sail is not particularly noble, rather it is in a specific position above the main sail and below the top gallant and royal sails. On a tall ship the word gallant is part of the nomenclature and has a special meaning. If there is a problem involving pressure, the pressure might be measured in PSI (pounds per square inch). More specifically it might be measured as PSIG or PSIA. PSIG is gauge pressure, that is, the difference between the inside pressure compared to the outside pressure. PSIA is the absolute pressure, that is, pressure inside relative to an absolute vacuum. While testing Apollo 6 one group of engineers wanted the PSIG at a certain level to test whether the vehicle could withstand the pressures that it would experience in space. Another group wanted to use the same oxygen ratio that was to be used in space in order to test the equipment that maintained the atmosphere. To accommodate both groups the test was run with the oxygen level specified for the PSIA level, but the test was run at the numerically equal PSIG rating. This oxygen-enriched pressurized atmosphere turned what should have been merely an expensive and embarrassing fire into a lethal tragedy. In math problems words like product, term, factor, even, function, limit, dense, almost everywhere, and many more have technical meanings. Don't ignore the significance of these technical meanings or you might get burned.

5.1.3 Summarize

Most people recommend summarizing the story problem as the first step. To create a summary you must find the essential parts of the problem. Typically a story problem is written as a sentence or a short paragraph which describes a scenario and includes quantitative information about the problem. Sometimes the scenario is simple enough that you can keep it in mind without a summary. If not, write a brief summary of the scenario. (Note that there's little point to doing this if your summary is as long as the original problem.) Your summary should exclude aspects of the scenario that aren't essential. Next, strip out the quantitave data and arrange it in a convenient form. Often this will be a list or a small table. Finally identify any descriptions of the relationships between quantities. Eventually these descriptions will have to be recast into mathematical language.

5.1.4 Picture

Draw a picture or graph of the situation. The picture could show the various objects in motion or the relative position of objects. The picture might be a graph of what a function might look like. It might be primarily geometric or it could be more realistic or highly stylized. The point of the picture is to help you visualize the problem in order to understand it.

5.1.5 Relevant quantities

As you read the problem you must identify the relevant quantities. Be specific. A distance, for example, must be measured between two points at a specific time. If items are in motion, clearly identify the time at which the distance is measured. If there are more than two points, you must clearly identify which two are involved. If the distance is between objects you must identify the point on the object. Note the difference between $d =$ distance and $d =$ distance between the rear of car A and the front of car B at the moment that car A completes the race.

Some students routinely assign names like x and y to the various quantities. That's fine, but I prefer to assign names based on the meaning, such as v for velocity and d for distance. If there are two distances involved, I might continue to use d to remind myself that these are distances, and then use a subscript so that I can tell them apart: d_1 and d_2, or d_c and d_p. Once you have identified the quantities it becomes easier to keep them in mind and to express the relationships between them.

In some cases the relationship between two quantities will be so easy to determine that it is easier to write the relationship than to use a variable for one of them. For example, rather than saying $r =$ radius of circle and $A =$ area of circle, you might find it more convenient to jump straight to $r =$ radius of circle and $\pi r^2 =$ area of circle. This is fine. In fact, if you later discover that you really wish that you had a separate variable for the area you may always come back and extend your initial definition: $r =$ radius of circle and $A = \pi r^2 =$ area of circle. You may also add more definitions as needed throughout the problem. For example, if you later realize that there are two circles involved, you might return and write $r =$ radius of circle (top) and $\pi r^2 =$ area of circle (top), $r_b =$ radius of circle (bottom) and $\pi r_b^2 =$ area of circle (bottom).

One advantage to finding all of the relevant quantities very early is that you can use r_t and r_b from the beginning, which is a little clearer than r and r_b. If you start off using just r and later add r_b, then you would have to go back and change all of your earlier r's into r_t's or you have to live with the slight asymmetry, and the problem of potentially forgetting which radius is meant by r. The corresponding disadvantage is that you might name many quantities that seem to be relevant but that never show up in the problem.

5.1.6 Key words

Some people rely heavily on the key-word approach to story problems. There are certain key words that routinely correspond to certain mathematical operations. For example the word "and" usually means either add or union. I don't like the key-word approach because the English language is so flexible that it is possible to make the key words convey the opposite of the ordinary meaning and it is possible to avoid the key words altogether. Here are some examples for which the word "and" takes on the standard and the reverse meanings:

1. Jack has two apples in his left hand *and* he has 6 apples in the basket in his right hand. How many apples does Jack have? Solution: $2 + 6 = 8$. Jack has 8 apples.

   Jack has two apples in his hands *and* he has some more in a basket. He has a total of eight apples. How many are in the basket? Solution: $8 - 2 = 6$ Jack has 6 apples in the basket.

2. Anne, Becky, Claire, and Denise are all car enthusiasts. Anne likes all cars more than 50 years old. Becky likes all luxury cars. Claire likes all cars that Anne *and* Becky like. Denise likes all cars that Anne likes *and* all cars that Becky likes. If A is the set of all cars that Anne likes, and B is the set of all cars that Becky likes, use set notation to describe C, the set of cars that Claire likes, and D, the set of cars that Denise likes, in terms of A and B. Solution: $C = A \cap B$ and $D = A \cup B$

Clearly you can't simply memorize that "And means add for arithmetic and union for set problems." The difficulty with setting up a story problem is to determine what mathematics corresponds to the verbal description. The best way is by matching the meaning of the words with the meaning of some mathematics. Nevertheless, sometimes key words are helpful, especially if you watch out for when they result in the opposite of the expected meaning. Here are some.

$+$: addition, all together, and, combine, more, sum, together, total

$-$: difference, distance, except, less, subtraction, take away

$\times$: area, by, multiplication, of, percent, times

$\div$: division, into, per, relative

$\cup$: also, and, combined, together, union

$\cap$: both, intersection, or

$=$: equals, is, makes

Various: each

5.1.7 Find the relationships

After identifying the quantities involved you need to find the relationships between them. These relationships are usually found in one of two ways: as common knowledge or as specified within the problem. Depending on the class you might be expected to know the relationships between things such as: length, width, perimeter, area, volume and so forth for rectangles, triangles, circles, cones, spheres, cylinders, etc; distance, time, and velocity; distance and acceleration; principle, time, and interest rate for compound interest; ratios and sums for geometric series; amount, growth/decay rate, time, and initial amount for constant relative growth or decay; profit, cost, and revenue; and so on. The better that you understand the basic and common relationships, the more easily you will be able to handle story problems that rely on these relationships. The second source of the relationships between the relevant quantities is within the problem itself. In this case the story problem gives a description of how the items are related.

5.1.8 Interpret the relationships

Express the relationships between the relevant quantities mathematically. In the case of known relationships this is usually as easy as writing down the formula that you remember (or have looked up), although you might have to change some letters depending on how you named the quantities in the problem. The more challenging problem comes when you have to create the formula from the description within the problem.

Translate: The most straight-forward way to do this is to translate the verbal description of the relationship into the corresponding mathematical relationship. This is simply done by reading the description, understanding what it means, and then writing the operation using mathematical shorthand.

Here's an example. "In three years my sister will be twice as old as me, but two years ago she was three times as old as me. How old are we?"

Name the variables.

M_n = My age now

S_n = Sister's age now

M_f = My age in three years

S_f = Sister's age in three years

M_p = My age two years ago

S_p = Sister's age two years ago

Describe the relationships between the present and the future/past.

$$M_f = M_n + 3$$
$$S_f = S_f + 3$$
$$M_p = M_n - 2$$
$$S_p = S_n - 2$$

Encode the future relationship.

$$S_f = 2M_f$$

Encode the past relationship.

$$S_p = 3M_p$$

This problem is now ready to be worked out. From here only algebra is required.

Sometimes people write down the equation that corresponds to the verbiage incorrectly. It is good practice to put in some sample numbers to verify that the equation corresponds to the description because it is easy to reverse an operation, for example, to accidentally write $2S_f = M_f$ where you need to write $S_f = 2M_f$. We have claimed that $M_f = M_n + 3$. Suppose that $M_n = 2$, then using the equation we find that $M_f = 5$. If I'm now two then in three years I'll be five. This formula is correct since the equation and the verbal description agree. We have also claimed that $S_f = 2M_f$. Suppose that in the future I'm 10 years old. Then by formula $S_f = 20$. This is correct since a twenty year old is twice as old as a ten year old.

Mathematical meaning: To help with translation it is valuable to remember the meaning of mathematical expressions. Most mathematical operations correspond to more than one type of situation, so it's helpful to be able to remember them all.

Addition: This gives the total when two groups of known size are combined.

Subtraction:
1. This is the inverse of addition— it undoes addition.
2. This gives the total of what's left when a group of known size is removed from another group of known size.
3. This gives the distance between two quantities.
4. It is adding a negative number.

Multiplication:
1. This is repeated addition.
2. This gives the area of a rectangle.

Division:
1. This is the inverse of multiplication—it undoes multiplication.
2. This counts the number of times that repeated subtraction can be carried out.
3. It is multiplying by a fraction.

Exponentiation:
1. This is repeated multiplication.
2. This gives the volume of a hypercube.

Root extraction: (Such as $\sqrt[n]{x}$.)
1. This is the inverse of exponentiation.
2. This is the number that has to be multiplied n times to get x.
3. This is the length of the side of an n-dimensional hypercube with volume x.

5.1.9 Units

Remember that you can't add apples and oranges. If you are having trouble determining what equation to use, check that the units make sense. You can only add or subtract quantities that are in the same units. When you multiply or divide quantities the units also multiply or divide.

2 apples + 3 oranges	This is trouble.
2 miles + 100 feet	This is trouble.
10,560 feet + 100 feet	10,660 feet
(6 miles per hour) × (5 hours) = $(6\frac{\text{miles}}{\text{hour}}) \times$ (5 hours)	30 miles
$C = \pi r^2$	This is trouble since the circumference C should be measured in something linear, like cm, but r^2 then has units of cm^2.
$C = \pi r$	This is trouble, although you can't tell from the units.
$C = 2\pi r$	This is the correct formula for the circumference of a circle.

5.1.10 Find the core

What makes the problem difficult? Why won't a simple answer work? Answer these questions and you'll understand the essence of the problem. Once you understand what impedes the solution you'll be better positioned to find a workable solution.

5.1.11 Identify the pattern

This method of understanding a problem often leads immediately to a method of solving the problem: following the pattern.

Often patterns can be more easily seen with the aid of a table or a graph. When it is difficult to determine the correct formula from the description consider making a table. Tables are especially useful when the description is given in inductive form. This most commonly occurs when one quantity changes with another in a regular way, such as the amount of money in an un-used bank account. It's easy to calculate the amount of money in the account after one year. Without the formula, however, the easiest way to get the amount of money after two years is to first calculate the amount after one year, and then use that as the starting point and repeat the calculation. Often the best use of a table is to have both a numerical and an algebraic column.

Consider the problem of calculating the amount of money in a bank account that earns 5% interest, compounded annually, 100 years after an initial deposit of $5,000. Notice that the standard description of earning 5% interest compounded annually, add 5% of the current amount to the current amount to get next year's amount, can be computed easily, but that it would take a long time to find the amount in the account after 100 years. However, since the process at each step is identical, there might be a pattern that can be exploited to jump straight to the end. If you don't already know the formula for this problem, it can be deduced readily by following the pattern through a table. Use D to represent the initial deposit.

time (yr)	balance ($)	formula
0	5,000.00	D
1	5,250.00	$D + .05D$
2	5,512.50	$D + .05D + .05(D + .05D)$
3	5,788.13	$D + .05D + .05(D + .05D) + .05(D + .05D + .05(D + .05D))$
100	?	?

This doesn't look very promising, but it becomes much more so when the formulas in the rightmost column are simplified.

Simplifying each formula in turn we see for year zero $D = D$. For year one $D + 0.05D = D(1 + 0.05) = 1.05D$. For year two we can apply the interest computation to the simple formula from year one to get $1.05D + .05(1.05D) = 1.05D(1.05) = 1.05^2D$. Using the simple year two formula to calculate the amount of interest in year three yields $1.05^2D + 0.05(1.05^2D) = 1.05^2D(1 + 0.05) = 1.05^3D$. Now we can extend the table using a simplified formula, and we can determine the pattern and thereby extend it to the hundredth year.

time (yr)	balance ($)	simplified formula
0	5,000.00	D
1	5,250.00	$1.05D$
2	5,512.50	1.05^2D
3	5,788.13	1.05^3D
t		1.05^tD
100	657,506.29	$1.05^{100}D$

5.1.12 Estimation

When faced with a long or difficult problem it is helpful to quickly estimate the solution. The estimate will be useful in two ways. First, the process of estimation can help you understand the problem and identify the difficulties with getting a precise solution. It might serve as a skeletal version of the solution or as a practice run. Second, you can use the estimate to evaluate your work. If your final answer differs too much from your estimate, at least one is wrong.

The best way to make an estimate varies dramatically depending on the type of problem. Sometimes you can apply a simple "rule of thumb" to estimate the answer. Sometimes you can calculate the answer using round figures. Sometimes you want to ignore aspects of the problem that make it more complicated, but that have a relatively small influence on the final result. Sometimes it is best to get an estimate that is simply close to the actual answer and at other times it is better to get an upper (or lower) bound, that is, an estimate that is certain to be larger (or smaller) than the correct result. (See section 5.1.13 for an example.)

5.1.13 Relevance of knowledge

Most of the time you subconsciously determine what informatioin is relevant to a situation without any problem. This is very fortunate since there are too many facts for you to consciously choose which to apply. Consider for example the classic algebra problem: "John can wash a car in 15 minutes. Karen can wash a car in 20 minutes. How long would it take them to wash a car working together?" If your subconscious relevance filter didn't work, you'd start solving the problem by noting "Aardvarks are small mammals from Africa," and several days later you'd reach "zymurgy is the chemistry of fermentation," and you'd still be on vocabulary so you'd have to keep going through the formulas. Nearly every one of these facts is completely irrelevant. Even facts about washing cars, such as "It's best to wash a car from the top downwards," or "It's important to rinse off the soap after washing the car." are irrelevant. It's critical to ignore irrelevant facts since it would take too long to evaluate their usefulness.

Nevertheless, your subconscious relevance filter creates a tremendous obstacle to solving a problem when it rejects critical information. To avoid this obstacle you sometimes have to consciously decide what information is relevant. To do this, identify the essence of the problem and which details are essential for finding the solution. Ignore other information. If you can't solve the problem consider widening your scope of what qualifies as relevant. This will give you much more useless information that you'll have to sort through, but you might find what you need amongst it.

For the car washing problem the essence of the problem is that both John and Karen are working at a certain rate, and therefore we can use the analogy of speed, time, and distance. Normally distance is equal to velocity times time (often written $d = vt$). We see now that John's speed is $\frac{1}{15}$ car per minute, and Karen's is $\frac{1}{20}$ car per minute. If we assume that speeds are additive in this context, then, while working together, they will work at a rate of $\frac{1}{15} + \frac{1}{20} = \frac{7}{60}$ cars per minute. In order to finish one car, we need to have just enough time, that is $1 = \frac{7}{60}t$, which can be solved for a time of $\frac{60}{7} = 8\frac{4}{7}$ minutes.

To illustrate a few other methods, let's check the reasonableness of this solution. If John and Karen worked together, both at Karen's speed, they should finish the car in half Karen's time, or 10 minutes. John works faster, so we expect our answer to be less than 10 minutes. Similarly we expect the answer to be larger than $7\frac{1}{2}$ minutes. Thus $8\frac{4}{7}$ minutes is plausible.

There are other ways to work the problem. One is to find a common interval of work. If they both start washing cars continuously for 60 minutes, then John will have just finished his fourth car and Karen will just finish her third car. If it takes them 60 minutes to finish 7 cars, then it takes them $\frac{1}{7}$ of 60 minutes to finish 1 car.

5.1.14 Sufficient Knowledge

Closely related to relevant knowledge is the question of whether you have sufficient knowledge to solve the problem. In math classes this can be problematic with story problems. In the real world it is more problematic. The information required to solve a story problem in a math class can be classified into five categories: information that is considered common knowledge, prerequisite information, information explicitly taught during the class, information provided within the problem, and information that must be looked up.

Common knowledge should be easy but in fact it can be problematic. Even well-educated people have stray little holes in their knowledge base. You should ask or look up knowledge as appropriate if you think that there is something that you're missing. Even if you know the information, you might not realize it is important.

In principle prerequisite information should be the easiest to find and use. It should be sitting in your brain ready to pop up on demand. This information should go through your relevance filter without any delay. However, many people do forget some of what they have been taught earlier. If you find that you have forgotten a lot of prerequisite material that shows up frequently you will find a review very helpful.

The information taught in class should be easy to find in your notes and/or the text. The problem is that if you are still learning the material you might not yet be able to use it well.

The information provided with the problem should be obvious. Typically with a math problem you should accept this information at face value, even if it isn't entirely true. Outside of a math test or homework problem, however, you should question any given information that seems incorrect. Many people make mistakes and it is better to fix an error than to propagate it.

Make sure that you understand the expectations for looking up information. If you are expected to look up the conversion factor for cups to liters, it might be tempting to also look up the formula for solving the whole problem.

5.2 Devise a plan

Here are several strategies that can be used to help devise a plan.

5.2.1 Backward questioning

Instead of solving a problem from beginning to end, try solving it from the end. Ask yourself, "What do I need to know in order to answer the question?" The next step is to ask yourself, "What do I need to know to reach that second last step?" Then ask yourself, "What do I need to know to reach that third last step?" Continue until the answer to one of your questions is something that you know how to determine. This method is nearly indispensable to the way that most people work out ϵ-δ proofs of limits in calculus classes.

5.2.2 Chunking

Break difficult problems into simpler sub-problems, called chunks. At first just make sure that solving each chunk would lead to a solution of the original problem. Next, choose one chunk at a time and solve it (perhaps by applying the method of chunking again). This method allows you to focus your attention on one part of the problem at a time so that you aren't distracted by too many details at once. A typical story problem in a calculus class might be: "Each month a store can sell 100 widgets for \$5.00 apiece. Whenever they raise the price by \$0.05 they sell 2 fewer widgets. It costs \$30 each month to handle widgets, plus \$2.75 for each widget sold. What price should be charged to maximize the profit? What price should be charged to just break even?" How could this problem be chunked out? One chunk is the formula for the number of widgets sold at a certain price. Another chunk is the formula for the revenue from selling widgets at that price. Another chunk is the cost of selling a certain number of widgets. This cost can be combined with the number of widgets sold at a given price to get the cost of selling widgets at a particular price. Profit can be calculated easily from revenue and cost. Algebraic methods can be applied to solve for a profit of zero to find a break-even point. Methods of calculus can be applied to find where the profit reaches its highest point. Regardless of whether you think of the chunks as the various formulas, or as the processes required to accomplish the various steps, it is much easier to determine what should be done by temporarily ignoring how the details of the major steps should be accomplished.

5.2.3 Simplification

Solve a related easier problem. The practice and extra insight can help you solve the origianl problem. Here's how:

1. Identify the factors that make the problem difficult.
2. Ignore these complicating factors and solve the problem without them.
3. Choose a complicating factor and put it back into the problem.
4. Determine the effect this complicating factor has on the problem. Why does it make the problem more difficult? How does it influence the solution? How do you modify your original solution method to take this new factor into account?
5. Repeat from step 3 until you have all the complications back in the problem and have determined how to handle each one.

Here is an example of the method at work: Solve problem 1: $x^2 + 2x = 4$. An easier problem would result if there hadn't been that $+2x$ on the left hand side. Eliminating it results in problem 2: $x^2 = 4$. The solution can be found simply by taking square roots on both sides, yielding $x = \pm\sqrt{4} = \pm 2$. Not much more complicated than problem 2 is problem 3: $(x+1)^2 = 4$, which is also simple. Taking square roots on both sides yields $x + 1 = \pm 2$. This hasn't been completed yet, but it can easily be solved by subtracting 1 from both sides to get $x = 1$ or $x = -3$. Superficially problem 3 doesn't look any more like problem 1 than problem 2 does. Notice, though, that the left hand side of problem 3 is $(x+1)^2 = x^2 + 2x + 1$, which now does look a lot like problem 1. In fact, it is only different from the left hand side of problem 1 by a constant, which can be fixed by changing the right hand side. This gives us a method to solve problem 1:

$$
\begin{aligned}
x^2 + 2x &= 4 \\
(x^2 + 2x + 1) - 1 &= 4 && \text{Prepare to simplify the left. Maintain equality.} \\
(x+1)^2 - 1 &= 4 && \text{Simplify the left.} \\
(x+1)^2 &= 5 && \text{Add 1 to both sides.} \\
x + 1 &= \pm\sqrt{5} && \text{Continue as before.}
\end{aligned}
$$

Now we get two possibilities: $x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$.

If you know the quadratic formula you can skip all this work and get the answer easily. However, people that don't know the quadratic formula have four options: give up, call an expert, pursue this or another line of reasoning that leads to the answer, and guess numbers and/or formulas at random until happenning upon the correct answer.

5.2.4 Probe

Propose a tentative solution to the problem. Examine the proposal and determine why it doesn't work. (If perchance it does work, then your problem is already solved.) Once you have a proposal in hand, and an understanding of why it won't work, consider whether and how it can be modified to get a feasible solution. Even if it can't readily be modified into a real solution, you will have gained some insight into what will work.

If necessary, probe the problem several times. It's possible that your probing will provide insight that will be helpful as you switch to another method. Another possibility is to use your new insight to propose an improved tentative solution.

The probe approcah is often inefficient but there are many exceptions to this. It can be a valuable initial approach since it can help verify that you understand the problem. Often one or two probes will help reveal the core of the problem. Frequently a few probes can provide insight that will be valuable when you switch to another method. It can work well when other alternatives are poor or when the total number of possible solutions is small. Finally, it is a method that works reasonably well regardless of learning or thinking style.

5.2.5 Outline

One good way to solve problems is to construct an outline of the process that would lead to a solution. Determine what needs to be done to get the solution. Put these things in the logical order from first to last. Check the outline to be sure that carrying out the specified tasks in the specified order should in fact yield the solution. Later go back and carry out each of the items in your outline. Note that with very complex problems you might apply the method of outlining several times over before reaching problems that can actually be solved.

5.2.6 Highways

Be selective in using the the highway method of solving problems since it is sometimes very powerful and sometimes nearly worthless. It is often effective for advanced math classes. If the problem presented can be solved using the methods of the recent material, then perhaps one or more of the theorems of the section will apply. Look up the theorems. Each one is like a highway that easily takes you from point A to point B. All you have to do is find a theorem for which the point A is close to what you are given in the problem, and for which the point B is also close to the conclusion that you need to draw. If you can find such a theorem, then all that remains to do is to figure out how to get from the starting point to point A and from point B to the finishing point. The theorem will do the rest for you.

While this style of thinking specifically targets problems in upper level math classes, it generalizes to a very practical approach to solving problems. Put in mind the things that can be accomplished routinely. Next determine how these can contribute to the solution of the problem. Finally, figure out how to connect the routine parts of your proposed solution to get the complete solution process. This approach tends to minimize the amount of non-standard, and therefore the most difficult, work required to reach the solution.

5.2.7 Knowledge diffusion

This is the least efficient of all of the major problem-solving methods. It is, however, the most powerful in that it can be used to solve more problems than any of the others. Start by identifying all of the given information. Next determine everything that you can conclude from the given information. Then start over and draw all of the conclusions possible from the given information augmented by your first set of conclusions. At each step you take advantage of your prior conclusions to give yourself more information so that you will be able to draw even more conclusions. As you start this method you restrict your conclusions to those that seem relevant. For instance, in the example from section 5.1.13 it is true that if Karen washed a car, and then John washed a car immediately afterward it would take 35 minutes. It's also true that if they both washed one car at the same time, John would finish 5 minutes before Karen. Neither of these facts seem helpful for determining how long it would take them to wash a car together, so you would hesitate to calculate them until you were so badly stuck that it becomes worth your while to search even the apparently irrelevant information.

5.2.8 Mulling

The brain has a surprising ability to mull or ruminate on a problem. That is, it can put a problem on the back burner and then suddenly think of the solution to a problem while not actively working on it. If you mull effectively, then you should take advantage of this ability. For best results choose one or two of the difficult problems to mull. (Most people can only effectively mull one or perhaps two problems at once.) Work these questions up to the point at which you get stuck. (If you finish a problem without getting stuck then choose a different problem to mull.) Once you're stuck move on and let your brain mull. Aperiodically you should review the problem or try to solve it again. Three factors influence when you should review the problem. First, you must review the problem often enough that it remains in your brain and isn't forgotten. Second, sometimes the solution doesn't come to your attention until you try to work the problem again. Third, if you review the problem too often your brain is more likely to lock onto an erroneous assumption, making it harder to reach the final solution. (See section 5.0.4.)

5.3 Carry out the plan

Sometimes the process of devising a plan flows naturally into carrying it out. At other times these are very distinct steps. Once you start to carry out your plan you might realize that the plan won't work or that it will be far more difficult than you had expected. If so you should consider whether it's better to go "back to the drawing board" and devise a new plan or whether to continue with the current plan. Make the decision on whether to start over doing something different based on how much work it will take to finish the job rather than how much work is already invested.

5.4 Evaluate the work

During the evaluation stage you should verify that the proposed solution is correct. Often you can decide whether the proposed solution is plausible. Often it can be checked explicitly by substituting it back into the original equation or the original description. You can always verify that the proposed solution has the right form.

Finally, unless you are confident that there is nothing valuable in your solution process, you should make a note of what you have learned while solving the problem. You might have learned new methods or new ways to apply old methods. With hindsight you might realize that there is a better way to solve the problem. Gain from the experience.

6 Diagnostic Tests

Businesses often have an 80/20 plan. They expect to earn 80% of their profits from 20% of their products. Likewise, you might prefer to read just a small fraction of this 56 page document if that would give you a large fraction of the benefits. If so, this section is for you. This section includes three diagnostic tests, one to determine your learning style, one for your thinking style, and one for special focus areas. These tests will help you to determine which parts of this whole document will help you most with your math class.

Your learning style summarizes how you learn information most effectively. The goal is to determine which forms of stimulation your brain responds to best, and then to emphasize these forms in your study. Your thinking style is based on how you process information. Your thinking style influences which are your most effective problem-solving strategies. Your center of attention determines which aspects of a message catch your attention. Knowing what you tend to notice and what you tend to overlook can help you work and study more naturally and can point out when you should conciously redirect your focus.

6.1 Learning style

There are three major learning styles: visual, auditory, and kinesthetic. For most people, one of these methods works better than the others.

Complete the table below in order to determine your dominant learning style. For each question place an `X` in the column which best describes your response.

	Visual Secion 7.1	Audio Section 7.2	Kinesthetic Section 7.3
It's easiest for me to memorize a formula if ...	__ I read and/or write it several times.	__ I repeat it aloud.	__ I tap it, point at it, or trace it with my finger.
Lectures are easiest to understand and remember if ...	__ the important points are written on the board.	__ the important points are spoken aloud.	__ I doodle or fidget during the lecture.
The best way to study is ...	__ to start by looking at lecture notes or the text.	__ to start by trying to remember what was said.	__ to take frequent breaks and move around.
While working a difficult problem I like to ...	__ visualize my notes.	__ talk myself through the problem.	__ think it through while pacing or doing something.
I find the following to be especially helpful:	__ graphs and diagrams.	__ a discussion with a classmate	__ any sort of hands-on activity.

The column in which you have the most `X`'s should represent your dominant learning style. If your secondary style isn't clear , go back and place an `O` beside each response that is a reasonable second choice. If you have a strong preference for your dominant learning style then you should emphasize that style in your study habits. Knowing about your secondary learning style and how to use it is also valuable if you are not highly polarized in favor of your dominant learning style or if you frequently find yourself in situations ill-suited to your dominant method. See the indicated section for study and problem-solving suggestions.

6.2 Thought patterns

Your preferred mode of thought is somewhere in the spectrum between linear and random. A linear thinker pursues thoughts sequentially. A random thinker jumps from one idea to another, sometimes before completing the first, without following an obvious pattern. On each line in the table below check the statement that best describes your attitude. The column with the most check marks will indicate your preferred mode of thinking.

Linear	Random
__ On a test my natural inclination is to answer each question before continuing to the next. It's hard to proceed to another question before finishing the current one.	__ On a test I often jump around to answer questions in the order of my own choice. It's natural to move on to another question when I get bogged down in one.
__ I prefer step-by-step instructions.	__ I find it frustrating to read directions that insist on doing this first and then that when it doesn't really matter.
__ When working a difficult math problem I find myself unexpectedly facing a dead-end and I don't know how to continue.	__ When working a difficult math problem I find that I usually don't start the calculations until I look ahead to other parts of the problem.
__ When writing an essay it makes sense to outline the contents before beginning to write.	__ When writing an essay I prefer to write parts of it first, and then verify that the content is arranged well.
__ My work for a long question on a math test is usually arranged in an orderly pattern, such as one or more columns of work.	__ My work for a long question on a math test is usually arranged with calculations scattered haphazardly around the page.

6.3 Attention focus

Your natural inclination is to focus your attention on things that are concrete or things that are abstract (or some point between these extremes). To test whether you prefer concrete or abstract thought, on each line below mark the statement with which you agree most.

Concrete	Abstract
__ After watching a movie I remember the actors, the scenes, and the details.	__ After watching a movie I remember the theme and the ambiance.
__ In math class the example is more important than the explanation of the example.	__ In math class, the explanation of the example is more important than the example itself.
__ When discussing a legal case that was on the news, I first want to know the details of the incident and people involved.	__ When discussing a legal case that was on the news, I first want to know the law and the principles behind the law.
__ I prefer numerical examples in math class to algebraic examples.	__ I prefer algebraic examples in math class to numerical examples.
__ Sample story problems best illustrate why math is important in the real world.	__ Explanations of how math is commonly used best illustrate why math is important in the real world.

6.4 Other diagnostics

If you would like to get the lion's share of the benefits while reading the mouse's share of this document then this test is for you. This test is especially helpful for people that by and large already have good study skills but nevertheless still suffer from specific weaknesses. For best results find the diagnostic questions with which you agree. Your highest priority will be those sections to which you are referred several times.

When I see a problem I know what to do but I can't actually do it.	See 10.2. Take the attention focus test to determine whether you are an abstract thinker. See 6.3.
I just need to get started. Once I get started I can finish the problem.	See 10.1. Take the attention focus test to determine whether you are a concrete thinker. See 6.3. Try the goal and backwards questioning methods. See 5.
I know what I'm doing. I just make a lot of careless mistakes	See 10.2 and 11.1.
I know what I'm doing. But I always lose a few points here and a few points there and it adds up.	See 10.2 and 11.1.
I only make a few mistakes but I make the same mistakes on lots of problems.	See 10.3.
I don't know where I lost most of my points. I haven't looked at my tests closely.	See 10.3.
During the test I get so nervous that I forget everything that I knew.	See 11.2.
It takes too long to memorize all that I need to know.	This is an odd complaint since once you learn the times tables your math classes require less memorization work than other classes. See 10.4 if you don't memorize effectively. See 6.3 if the problem is with the type of material that you are trying to memorize.
I know how to do it at home. I just get too nervous during the test.	See 11.2.

It seems easy when I see it in class, but then when I try it myself I can't do it.	Unfortunately this common complaint is not precise enough to diagnose the real problem. If you are waiting too long before starting the homework then try hard to start the homework as soon after class as practical. If you aren't learning the material well see 3.7, 3.10, 3.11, and 3.12. Routinely do your homework with a study group (see 3.8) or in a math help center where you can get help. Apply active learning methods in class. Try to work the problem faster than the professor at the board, or at least to predict what his next move will be.
Some of the test questions weren't like the homework and I didn't know what to do with them.	See 2.2 for information on what to expect. Linear and concrete thinkers are especially susceptible to missing some of the conclusions. See 6.2 and 6.3. See 4.8 on how to respond when this situation arises and 5 for suggestions on how to attack the unexpected problem.
I missed questions that were similar to the homework problems.	See 3.11.
The wording throws me off.	See 5.1 and 4.9.
I could have done better but I ran out of time.	See 4.1.
I knew most of the material, but the test covered the things that I didn't know.	The sure solution is to learn all of the material. If you're unwilling to do that then indicate in your class notes everything that was emphasized in class or on review sheets and give those things a high priority. See 3.4.

7 Learning In Style

7.1 Visual learning skills

Clearly visual learners should incorporate visual cues during study and problem solving. For visual learners, these suggestions should be great additions to those found throughout this document.

Reading the text (beyond section 3.5)

- Write in your text. You will forfeit resale value if you write in your text, but you'll learn much more effectively if you mark it up so you can see what you are trying to learn [1].

- Write notes in the margin.
- Use a highlighter (or better yet, several colors of highlighter, each with a distinct use) and underline things to draw your attention to the main points.
- Draw pictures or graphs to illustrate the ideas.
- If your notes don't fit in the text, attach extra pages, either with sticky notes or even by sticking a whole sheet into the book.
- Draw arrows to indicate things that are related. If it relates to something on a different page, point the arrow at a page number. If possible, draw a picture or write a short summary of what the reference is to.

Using notes (beyond section 3.7)

- Your notes are very important to you. Be sure to take careful notes.
- Rewrite your notes several times. This is an effective learning strategy for you.
- Annotate your notes as described above for the text.
- Add extensive pictures and diagrams to your notes.

Mnemonics (beyond section 10.4) Focus especially on the flash cards, disapearing prompts, spider diagram, and picture methods. You should also find the pattern, acronym, and mental journey methods helpful.

Picture (beyond section 5.1.4) Since your learning style is visual, drawing a picture or diagram is probably the most effective method for understanding problems.

Pattern (beyond section 5.1.11) This method of understanding and solving problems lends itself to visualization.

7.2 Auditory learning skills

Auditory learners should emphasize listening and speaking. The suggestions below are especially for auditory learners.

In class (beyond section 3.6)

- Look for a note-taking partner. This could be very helpful.
- Choose a good seat. Sitting near the front of the class results in fewer distractions and a better chance to hear. Since paying attention to and hearing the lecture are critical to your success, be sure to sit where this will be easiest for you [1].
- Ask questions in class. Professors tend to respond orally, giving you an advantage [1].

Using notes (beyond section 3.7)

- Written notes aren't as helpful to you so put more effort elsewhere.
- Use shorthand so that you can write your notes faster and save time to listen and think.
- If you don't need it in your notes, don't write it down. This includes what you already know and what you can figure out on your own.
- Record the lectures and then play them back to yourself. (Please ask your professor for permission to record lectures. Nearly all professors are willing.)

- Read your notes aloud when you use them.

Group work (beyond section 3.8)
- You are likely to find discussions with other classmates very helpful.
- Ask classmates to describe how to solve problems when you meet.
- Try to meet people in person or call on the phone rather than sending e-mail and text messages.

7.3 Kinesthetic learning skills

Kinesthetic learners should try to move while studying and solving problems. There are three ways to move as you study. The first is to move a little while studying. Figure out how much you can move without distracting people around you. You should be able to do things such as move your hand and fingers, move your feet, and adjust your posture. Carrying out these motions as you study will help you concentrate. The second is to take breaks from studying to move. During your short and fequent breaks you should move extensively. These first two methods use motion to activate the brain but don't specifically use motion to facilitate learning the material. If you use these methods you'll want to supplement your study techniques with methods appropriate for your secondary learning style. The third way to move requires more creativity on your part. Rather than merely moving while you study, use motion as a specific learning tool.

More quick hints (beyond section 3.2) Be sure to work problems and even extra problems. You'll profit from simply pushing your pencil through the motions required to solve the problem.

Mnemonics (beyond section 10.4) The spider diagram method can be helpful [1]. Feel free to revise the mental journey into a physical journey. The pattern method can be made active if you move your hand down the pattern. Use flashcards with exagerated movements. You can modify the song method by including hand motions or enough choreography to turn it into a dance method. (Small hand motions have the advantage that you could repeat those during a test if you need to.)

Kinesthetic study suggestions
- Take frequent study breaks. After studying for fourteen minutes, take a one minute break for physical activity. Toss a tennis ball halfway to the ceiling. Get a sip of water from the fountain on the next floor. Do a few jumping jacks or squat thrusts [1].
- Fiddle as you study. Roll a tennis ball back and forth on your desk. Fold paper into tiny squares. Doodle. Pace. Twiddle your thumbs.
- Develop hand motions to associate with the formulas, theorems, and equations. For example, you might tap your finger lightly on each term of an equation, or on each variable in a formula. You might move your finger in an arc following the path of a variable as it gets multiplied according to the distributive principle, or as it is divided onto the other side of an equation.
- Develop a personal sign language. Once you learn a certain motion or position of your hand for each arithmetic operation, you can describe a formula or procedure with a sequence of hand motions.

- Write notes on small pieces of paper so you can work on memorizing things while jogging, riding your bike, playing football, or other physical activities [1].
- Manipulate things in accordance with the arithmetic operations that you're doing. Remove a few paperclips from your pile when you subtract. Divide a pile into pieces when you divide, and so forth.
- As appropriate write notes, homework, and study guides with large print so that your hand moves as you write. Better yet, make posters or write things on a black board so you can move your entire arm.
- Walk a path that spells out a formula [1].
- Act out solving a problem. Imagine that x has been captured and is imprisoned in a castle. With each mathematical step of solving for x, act through the process of taking the captive a step closer to freedom. You might choose to treat distributing (to get x out of parentheses) to be equivalent to removing shackles, moving all the terms with x to one side of the equation to be the equivalent of walking out of a cell. Perhaps dividing by a coefficient is crossing the moat (or barbed wire if you prefer a modern prison).

Perhaps you don't like these suggestions because they seem silly or aren't widely applicable. If so, you're welcome to develop better alternatives. On the other hand, which is sillier, working problems on poster-sized paper, writing large, making hand motions, and moving around; or squandering ghastly amounts of time to learn the material slowly.

8 Using Effective Thought Patterns

8.1 Linear thoughts

Typically linear thinkers readily accept rigid arithmetical and algebraic processes but have more trouble with those aspects of mathematics for which experience is more valuable than procedure (such as factoring polynomials), or for which the starting point seems to come out of the blue (such as $\epsilon - \delta$ proofs).

Study strengths: You are good at learning the procedures for solving problems. You readily take good notes. These study techniques are effective for you: outlining, using patterns, flashcards (especially when kept in a systematic order), and the mental journey. (See 10.4.) Include with your notes a summary of each method that you cover, written in the form of step-by-step instructions.

Study weaknesses: It's easy to miss secondary relationships between things. To spot these make an outline with an organizational principle other than what's familiar or make a spider diagram, perhaps with an extra emphasis on the relationships. (See 10.4.) Make sure that you can work problems without the hint of knowing from which section the problem comes. (See 3.11.)

Problem solving strengths: Your strong problem solving methods include: summarizing the information, looking for a pattern, outlining a solution, and using the highway. With practice, estimation, backwards questioning, simplification, and chunking will also be very valuable methods. (See 5.)

Problem solving weaknesses: It might take a lot of practice to add mulling to your arsenal of effective methods. Use knowledge diffusion only as a last resort. (See 5.)

Test-taking tips: Force yourself to postpone spending time on difficult questions until you have already worked the simple ones. (See 4.1.)

8.2 Random thoughts

Typically random thinkers dislike the algorithmic processes and the common mandate to show work but respond well when ideas or calculations have to be left hanging pending some other result. Basic mathematics requires linear thinking. At this level random thinkers must apply the self-discipline to use linear thinking or find mechanisms to cope with the mismatch between thought style and the mathematical demands. On the other hand random thinkers are well-positioned to excel in mathematics once they get beyond these initial phases of mathematics.

Study strengths: You are good at integrating what you learn into a cohesive understanding of the whole. These study methods reinforce your natural strengths: spider diagrams, flash cards, finding patterns, and using pictures. (See 10.4.)

Study weaknesses: You naturally turn out disorganized class notes. You will benefit greatly from organizing your notes, so take the time to do so, even if it's painful. (See 3.7.) You rebel against the procedural aspects of mathematics. It is difficult to state a principle in your own words precisely and concisely. Try the acronym, mental journey, or pattern memorization techniques to help learn sequential information. (See 10.4.)

Problem solving strengths: Your strengths include mulling, finding the pattern, estimation, chunking, and knowledge diffusion. With practice you can also excel at finding the core, simplification, and using the highway. (See 5.)

Problem solving weaknesses: You can benefit greatly from mastering backwards questioning and outlining. (See 5.)

Test-taking tips: Verify that you have completed every problem while you still have enough time left to complete any that you might have skipped.

9 Focusing Your Attention

9.1 Concrete thinkers

Concrete thinkers prefer the specific over the general. Examples are more meaningful than explanations. Anecdotal evidence is more persuasive than statistical evidence. A hypothetical scenario is easier to discuss than an abstract principle.

Study strengths: You learn from example, so keep examples handy in your notes. You like things that you understand well and in detail. Therefore you should create tables or pictures to illustrate patterns in a concrete way. Make flash cards with specific examples rather than generic examples. The acronym and mental journey method of memorization work well for you. (See 10.4.) Be sure to emphasize examples in your notes. Give examples priority over the explanations.

Study weaknesses: Sometimes you have trouble remembering the general principle that an example represents. If so make a special effort to learn the general principle as well as the example. You can have trouble if the class examples don't cover all of the important cases for some principle. When this happens you'll have to add to your notes an example that will illustrate each case. (For example, when learning about using the quadratic formula, you might need three examples, such as $0 = x^2 + 6x + 4$, $0 = x^2 + 4x + 4$, and $0 = 3x^2 + 2x - 1$ to illustrate the cases that have 0, 1, and 2 distinct real roots. You might even prefer to have an example for each type of solution: integer, reduced rational, non-reduced rational, and irrational.) You also need to include exceptions to your formulas. Instead of writing "$x^0 = 1$ for $x \neq 0$" you might need to write "$x^0 = 1$ but $0^0 =$ undefined."

To get through the more abstract classes you'll want to take abstract concepts, such as set, function, derivative, and ring and become familiar enough with them that you'll be able to treat them as concrete objects. There is a lot of power for you in developing the habit of creating concrete models of abstract principles, especially if you also understand the significance of the models and examples. You might have to constantly remind yourself of the principle behind the examples that you learn.

If a section seems unimportant, then find a specific application (such as a story problem) using the material and include that in your notes. (If a section has no direct applications then make a note of an application that relies on the section.)

Try to find a specific rule or list of rules to express the principles in each section. Explicitly list these rules.

Problem solving strengths: You'll understand a problem much better if you draw a picture and clearly identify the goal. You'll also spot patterns better if you create a table of example data. Be sure to verify that you have the correct equations with sample solutions. Try the simplification method. It can generate an example of a near solution for you. With practice the method of chunking can be very effective for you. (See 5.)

Problem solving weaknesses: You might struggle with problems that don't resemble examples that you know. In this situation try listing the principles that seem to apply. You'll need to master the skill of understanding the meaning of examples including how to generalize the example and what other situations are similar.

9.2 Abstract thinkers

Abstract thinkers focus on principles and patterns rather than specific individuals or incidents. A clear explanation of principle is more meaningful than an example.

Good Flash Card For Concrete Thinker

Front of card	Back of card
Find the slope between the points $(3,5)$ and $(6,-1)$	$m = \frac{5-(-1)}{3-6} = \frac{6}{-3} = -2$

Good Flash Card For Abstract Thinker

Front of card	Back of card
Find the slope between the points (x_1, y_1) and (x_2, y_2)	$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$

Study strengths: You quickly grasp the main idea of a lesson. You do well with the pattern, picture, and probably the spider diagram or outline methods. (See 10.4.)

Study weaknesses: Once you understand the major points you might feel impatient with the need to learn the rest of the material. Remind yourself that the details are important (Recall that the $1,700,000,000 space shuttle Challenger with 2,000,000 moving parts exploded because two rubber seals leak when it's cold.) and then apply the self-discipline to finish. If you have trouble remembering details try flashcards, acronyms or working extra problems. (See 10.4.)

If a portion of the material seems pointless, make a note of the goal of that material. Indicate what type of problem can be solved with this material.

Problem solving strengths: You are good at determining the relevance of knowledge and at finding the general idea of a solution. You do well with the pattern-finding, simplification, outline, backwards questioning, chunking, and highway methods of problem solving. (See 5.)

Problem solving weaknesses: You have trouble with subtle problems and those that hinge on tiny details. You might have trouble implementing your plan. You might have trouble modifying standard methods for situations that are a little atypical. The main correction is to proceed with diligence and self-control. You might be able to sweeten the drudgery by thinking of the specific problem as an illustration of the variations in the basic principles used to solve similar but simpler problems.

10 Specific Study Skills

Perhaps by definition the average student can succeed using normal means. Routine study suggestions are given in section 3. Some students, especially those with a lopsided set of talents, might have to use extraordinary means to master one aspect or another of the material. The suggestions below are intended for students who find the routine suggestions inadequate.

10.1 The big picture

- After each class write a paragraph explaining how the new material relates to the old material.
- Create a table of all of the formulae discussed in class. Can the formulae be organized to form a coherent pattern?
- While doing work, answer the questions "Why is this true?" and "Why is this helpful?" at each step. This will force you to constantly remind yourself of how the detail of each step fits into the larger goal of solving the problem.
- Explicitly write the goal of each problem that you solve near the problem. If it helps with your motivation, write a real problem (that is, a story problem) to remind yourself that you are practicing solving a type of problem that occurs in the real world.
- Sometimes it would appear that either of two steps are reasonable to apply, but in fact only one is helpful. When this happens, write down an explanation of how to know which step is correct.
- Write down a number of goals to obtain, rather than merely equations to solve.
- Learn the meaning of the computations. Determine what they mean taken as a whole.

10.2 The details

- Work more practice problems. Remember that ultimately you have to do the work yourself, not just understand someone else's work. Choose practice problems from prior sections and even prior classes if you have trouble with that type of work.
- Write neatly to avoid misreading your own work.
- Read and write with care. Take enough time to get the details right.
- Identify where you make the most errors. Check your work of this type.
- Try to add meaning to the details by thinking of the goals that the details accomplish. For example, don't simply make the whole problem "Solve for x." Make the first problem "Get all the x's on the right hand side." Then start over with a new problem: "Isolate the x's." Even though the pieces are small, they might act as if they are the big picture if you treat each one like the big picture in turn.

- Create a table of every formula discussed in class. Can they be organized to form a coherent pattern?
- Write formulas down correctly. Include the restrictions. Write "$a^0 = 1$ for all real numbers a except $a = 0$", rather than "$a^0 = 1$".
- Determine the meaning of the computations rather than just the mechanics. It's a lot easier to remember a meaningful procedure than to memorize the details of a meaningless one.

10.3 Learn from your mistakes

Some students feel embarrassed by their mistakes. Don't. Mathematicians work in pencil, chalk, or dry erase markers. We don't hate ink. We simply can't work with indelible writing implements. We expect to make mistakes and therefore plan to correct them as easily as possible. Everyone makes mistakes, especially people that are learning.

- After a test find the correct solution to each of the problems that you missed. Identify what you did incorrectly and determine why. Work the problem again, correctly, to reinforce in your mind the correct way to solve the problem. Although it may seem like closing the barn door after the horses have gone, it is likely that the issue will come up again at some point.
- Review your homework. Correct any mistakes. You might have to work several problems in order to train yourself how to work the problems correctly, especially if you have already worked several the wrong way or if you learned something wrong in the past.
- If you discover whole patterns of errors look for the underlying cause and correct it.
- Your professor or members of your study group can help you with your questions. Be persistent. If someone tells you to work a problem in a certain way, and you don't understand why that way is correct, or what the difference is between that way and another way, ask.

10.4 Mnemonics: The art of memorizing

Mathematics requires less memorization than most other subjects. Despite spending hours trying to memorize the order of the numbers (Do you remember trying to count to 100 when you were little?), the addition table, and the multiplication table, higher levels of mathematics relies much more on experience, understanding, pattern recognition, and analytical skills than on memorization. Nevertheless, you do have to memorize some information. If this is difficult for you identify a few helpful techniques. Here are some suggestions.

Understand: It is much easier (and more helpful) to memorize material that you understand. If possible, determine what the material means and why it is important before trying to memorize it.

Practice: The best way to learn, and remember, how to do mathematics is to practice. If you aren't assigned enough homework to actually learn and memorize the material, then work extra problems. It also helps to work a few problems as soon after class as possible, and then a few more problems of each type each day for a week. After that you should work one or two problems of each type two or three times a week for the next two or three weeks. After that you should be able to remember how to work the type of problem by working one example every week or two. Although this looks like a lot of work, in practice you can often do a lot less. In many cases the problems that show up early in the class recur later as parts of bigger problems. If so you won't have to work separate examples.

Flash cards: Flash cards serve well to help memorize or to help review. Write the question or prompt on one side and the answer on the other. For best results put only one idea on each card, even if that means you'll have a huge stack of cards [1]. Read the prompt and try to remember the answer before checking. Experiment to see whether you do better with frequent short bursts or a few long sessions. Determine whether you should focus on a few cards at a time, or divide your attention amongst them all.

Acronym: Choose the primary letters of the word or phrase that you need to memorize, and turn those into a sensible word or phrase. I can remember the order of the 8 primary planets because "Mary very easily makes jam Saturdays until noon." (Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune). Some students like the word SohCahToa which stands for "sine: opposite over hypotenuse, cosine: adjacent over hypotenuse, tangent: opposite over adjacent." You can make up a story or word whenever you need one.

Song: If you find it easy to learn the lyrics of songs, then you might try fitting things that you want to memorize to a simple tune or to modify the slogan for a commercial.

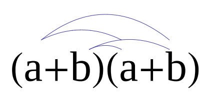
Disappearing prompts: Write out the entire formula or theorem that you want to memorize. After reading through it a few times, cross off a small part (just a word or two from a sentence, or a single symbol from a formula). Once you can state the whole thing even with a small part missing, cross off another part.

Mental journey: The ancient Greeks practiced very effective memorization techniques. Their skilled statesmen and actors only needed two or three days to memorize a two or three hour monologue. Many would be able to remember these monologues years later. Likewise couriers were expected to be able to repeat, verbatim, a message of a few paragraphs a few days after hearing it once. One method that they employed was to visualize a journey through a familiar place. Commonly they chose a walk through their own home, or between two places that they visited daily. On this journey they would routinely notice approximately 50 stations. Each station was a specific item or place found on the journey. Once the journey with its stations was learned, they would memorize new facts by imagining the journey and visualizing things that they associate with each station. With practice they could rapidly memorize an ordered list of images. Since you can visualize a noun, event, or action, either simple or complex, this gave them a tremendous ability to memorize things. This method becomes stronger with practice and thus becomes very effective if used regularly.

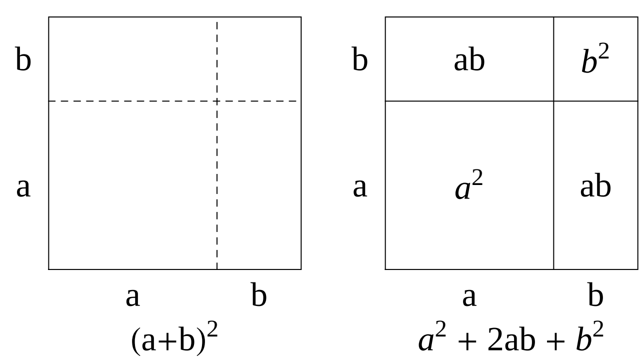
Word games: You can probably find hundreds of words that rhyme or alliterate with the ones you want to learn. Combine some of these in a memorable way so that you can remember the words in question.

Story: Invent a little story to go with the facts to keep them in order. Most people can hold approximately seven independent words in short-term memory. A story can create a link so that the facts are no longer independent but form a sequential chain so that each word can be drawn into memory one at a time.

Picture: Draw or visualize a picture. Most people memorize images more quickly than words, and visual memories are more persistent than auditory memories. Many students forget that $(a+b)^2 = a^2 + 2ab + b^2$. They forget the middle term. How could they remember this? One option is to always work out $(a+b)^2$ using the basic principle of the distributive property. The second option is to directly memorize the answer, perhaps using flashcards with $(a+b)^2$ on the front and $a^2+2ab+b^2$ on the back. The third option is to get a good acronym, such as FOIL, so that you remember to multiply the First terms, then the Outside terms, then the Inside terms, and finally the Last terms from each factor. The fourth option is to visualize arrows showing the multiplication:



The fifth option is to visualize the actual motion. Watch the first a move over to multiply the second a, and then watch the first a move over to multiply with the second b. Then the first b will also move over to multiply with the a and then the b will move over and multiply with the b. The sixth option is to visualize the problem using area, as shown below.



Spider diagram: A spider diagram shows the relationships between various concepts. To create a spider diagram, start by putting the main idea in the center of the page and drawing a circle around it. Arrange the secondary ideas around the primary idea. Then circle each secondary idea and connect it to the primary idea with a line. Next add the tertiary ideas, circle them, and connect them to the related secondary ideas. There are several variations to the basic spider diagram. You can use color or different types of circles to mark different types of ideas, such as to distinguish examples from principles, or old ideas from the new ones. You can use different types of lines to indicate the types of relationships between the ideas. You can also connect all the ideas that are closely related, rather than connecting each item to only one more important item.

Analysis: Analyzing information aids in memorizing it. One way to force yourself to analyze the information is to organize it according to different organizational principles. For example, outline all the material that will be on the test arranged in the order that you covered it in class. Then go back and make another outline arranged by usefulness, and another arranged according to applicability, another arranged by topic, and another arranged by requisite skills. If you don't like outlines, use spider diagrams, flowcharts, or some other way to arrange the material. The point is that you will learn the material by putting it into order from a number of different perspectives.

Pattern: Identify a pattern and use it to extend what you know. For example, if you consistently forget the meaning of x^0 or x^{-1} you can create a table of what you do know and extend it to the other exponents. First we switch from x to 2 in order to have a concrete example for our pattern. Then we create a table (below left) indicating what we know. Then you extend the table (below center) to clarify the pattern: moving down the table exponents are increased by one in the left column and the numbers in the right column are multiplied by two. Then we can reverse the pattern (below right) so that we can move upwards from the completed portion of the table into the portion that is not complete. Instead of adding one to the exponent we subtract one when moving up the table, and instead of multiplying by two we divide by two when moving up the table.

What's Known	
2^{-2}	?
2^{-1}	?
2^0	?
2^1	?
2^2	4
2^3	8
2^4	16
2^5	32

Downward pattern			
exponent			number
	2^{-2}	?	
+1	2^{-1}	?	
+1	2^0	?	
+1	2^1	?	
+1	2^2	4	
+1	2^3	8	×2
+1	2^4	16	×2
+1	2^5	32	×2

Upward pattern				
exponent			number	answer
−1	2^{-2}	?	÷2	$\frac{1}{4}$
−1	2^{-1}	?	÷2	$\frac{1}{2}$
−1	2^0	?	÷2	1
−1	2^1	?	÷2	2
−1	2^2	4	÷2	4
−1	2^3	8	÷2	8
−1	2^4	16	÷2	16
	2^5	32		32

Consider one more example, which helped me when I first took trigonometry. Would you rather memorize the values of the sine and cosine as they are printed in columns two and four below, or as they are printed in columns three and five?

x	$\sin(x)$	$\sin(x)$	$\cos(x)$	$\cos(x)$
0	0	$\frac{\sqrt{0}}{2}$	1	$\frac{\sqrt{4}}{2}$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1	$\frac{\sqrt{4}}{2}$	0	$\frac{\sqrt{0}}{2}$

10.5 Prepare for the test

- Verify that you are ready for the test. Use a practice test to spot problems and then boost your confidence. See section 3.11.
- Pretend that you will be allowed one page of notes for the test. Prepare the page of notes. Go through your class notes and write the most difficult and most important things on your

one page "cheat sheet." Then spend the last few days before the test studying from the "cheat sheet."

- Review all of your class notes and also skim through the text. Stop at every point that you didn't understand. (These should all be marked.) Verify that you can answer your question now.
- Get a sheet of paper and write down each major type of problem that you expect to see on the test. For each problem outline the steps that you go through to solve the problem. If you don't know how to complete a step that shows up, be sure to review it.

11 Specific Test-Taking Strategies

11.1 Careless errors

- Write neatly to avoid misreading your own work.
- Read and write with care. Take enough time to get the details right.
- After working a problem re-read the question and your final answer. Make sure the answer makes sense and actually answers the question.
- If you make specific errors generate a personal error-proofing strategy. For example, if you routinely lose minus signs when using the distributive property, consider actually circling the factor, with its sign (whether negative or positive), and drawing an error from the circle to each term of the factor in parentheses. Another option would be to develop the habit, each time you complete a line of work that includes a distribution, to immediately go back and check each plus and each minus sign and verify that they are all correct.

11.2 Math anxiety

For most people the single most important thing to do is to prepare for the test. If you know the material then there is nothing that you need to be nervous about. Once you eliminate the root cause for the anxiety you should be able to control any residual anxiety through the various ways to relax or psych yourself up. You need to resolve math anxiety with a three and a half-pronged approach.

1. Eliminate genuine causes for anxiety.

   Come well prepared. If you know the material then there will be no reason to fear.

2. Eliminate false causes of anxiety [7].

   - You're stupid because you make stupid mistakes.
     Many smart people make stupid mistakes. In fact, many "stupid mistake" are caused by rushing too fast, emotional distress, or misunderstanding some mathematical concept.

- You're stupid because you can't learn mathematics.

  Math can be one of the most challenging academic fields. Nevertheless, armed with determination and effective study methods nearly every college student can succeed with their first-year math classes.

- You fear that you're about to lose your scholarship, your good GPA, your chance to get into your preferred program, and your whole future.

  Get a grip. One bad question, one bad test, even one bad class won't destroy your future. Worry about the future can only have two effects: It can reduce your mental capacity or it can prompt you to work harder (See 2.1.), work more effectively (See 3.), and seek the help that you need (See 3.12.).

3. Identify and eliminate those things that make you uncomfortable. (See section 4.8 for more suggestions.)

4. Provide those things that make you comfortable.

   This is really the same as eliminating what makes you uncomfortable, but viewing the problem from both perspectives will help you identify more things to do.

   Make your equipment comfortable. For example, if you don't like your calculator because it has too many buttons on it, but you can't use your simple one because it has no trig capability, then bring both calculators to the test. You can use your favorite one for everything except the trig.

   Another option is to work your homework with your fancy calculator and learn how to use it well enough that you become comfortable with it. Determine the limits that make it reach an error. What is the largest number it can represent? What is the smallest positive number it can represent? Determine how it handles the order of operations. What answer do you get from $1 + 2 = \times 3 =$, and from $1 + 2 \times 3 =$. The better you understand your calculator, the less you have to worry about not being able to make it do the right thing.

   Another tack is to personalize your calculator. Put a sticker on it or draw a picture on it. Paint it another color. Turn it into a toy. Perhaps some of the numbers look like letters when you turn your calculator upside down. If so, write a message for your friend, and then invent a computation that gives the message as the answer.

   Likewise, if you don't like your pencil you can get a grip to make it more comfortable. Ask your professor whether you may use a pen with your favorite color ink.

   Make the room comfortable. If the view is distracting try sitting in a corner or ask for permission to turn your desk around backwards. Come in before the test, draw a fun picture on the blackboard, and ask the proctor not to erase it.

References

[1] *Math Study Skills* by Alan Bass, 2008

[2] Russell Hendel of Towson University in *MAA Focus*, May/June 2008, p. 23

[3] Credited to Ms. Hipke, these hints were passed on to me by Leona Mirza, North Park University.

[4] *Patterns of Problem Solving* by Moshe F. Rubinstein, 1975

[5] *Success in Mathematics* a document produced by Saint Louis University and available at http://euler.slu.edu/Dept/SuccessinMath.html

[6] Suggestion offered by Greg Tollisen, Occidental College

[7] Wild about Math suggestions for getting past stupid mistakes, available at http://wildaboutmath.com/2007/11/09/how-to-get-past-stupid-math-mistakes/

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